BACK

TAYLOR SERIES EXPANSIONS

by Bärbel Barzel

My presentation shows a possibility, how to teach pupils of 16–17 years old (in Germany the last class before school-leaving examination) in "Taylor series expansion" with DERIVE.

"Taylor series" are seldom taught in our schools because they are quite difficult and abstract for pupils when taught the "classical" way. That's understandable but it is a pity, that this theme is so seldom offered at schools. Taylor series can bring approximation problems, series expansions and their importance close to the pupils. There is for example the question, "how does the calculator compute sine"?, which can show the importance of those subjects. That's why I tried to work out a more attractive way (for pupils as well as for teachers), how to teach Taylor series.

1. Preliminary knowledge

The pupils must have knowledge in the following subjects:

derivations

transcendent functions and their derivations

main formula of differentiation and integration

partial integration

induction

2. Introduction

The above-mentioned question, "how does the calculator compute sine?" is the very first beginning of the sequence to motivate the pupils to try and find a possibility to approximate the sine-function.

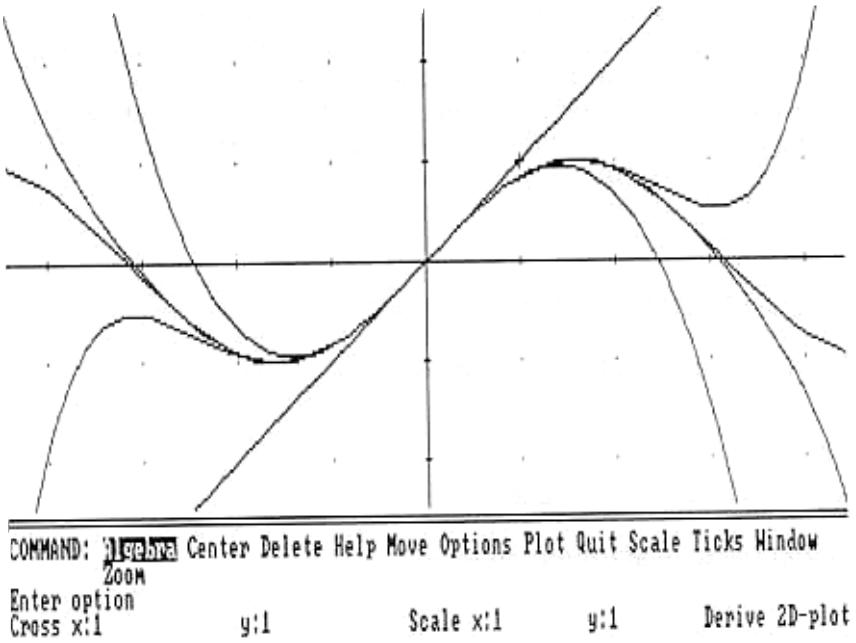
In the first lesson the pupils get a work-sheet with the following instructions:

2.1 Type the function f(x)=sin(x) with DERIVE and make use of the command "Calculus Taylor".
You are asked for "degree" and "point". Stay on the point 0 and change the degree from 1 to 7.
Let the curves to all the expressions of the algebra window plot out in the graphic window.
What can you realize?

2.2 Write the expressions of the Taylor expansions to the sine-, cosine- and e-function down into a table.

Can you find out a general rule?

The two instructions agree with the two main functions of DERIVE.
The first one question uses the graphic part of DERIVE. The pupils have to realize the main sense of Taylor series expansions: They see at the screen (picture 1), that the curves to the Taylor polynomials approximate the curve to the sine-function in the surrounding of the point 0. The pupils can see: the higher the degree of the expansion the better is the approximation.



picture 1

I observed that really every pupil finds out this main sense of Taylor expansions. They kept this in mind, because it was their very first introduction and they worked it out themselves at their individual learning pace.

The second instruction of the work-sheet (see above) uses the algebraic functions of DERIVE. The pupils are asked to fill out the table by using DERIVE. That's to say, they had to find out the Taylor polynomials from the first to the 7th expansion degree of sine-, cosine- and e-function. Looking at these special expressions they have to find out the pattern for the general formula of Taylor (or more exactly of the formula of Mac Laurin, because in this first stage of the unit we rest on the expansion in point 0).

To help you imagine the ideas and thoughts pupils had during this period, I give you some free quotations from the pupils:

> "In the denominator must be a factorial!"
> "This factorial agrees on the power of x!"
> "There are terms, you can't find by the expressions to sine or to cosine – they must be 0, so that they are not shown!"
> "Some expression are 0 by sine or cosine and not by e-function – similiar to the derivations...also the change of the sign agrees to the derivations.."
> "If the signs would be a bit other wise, you could build the terms to the e-function

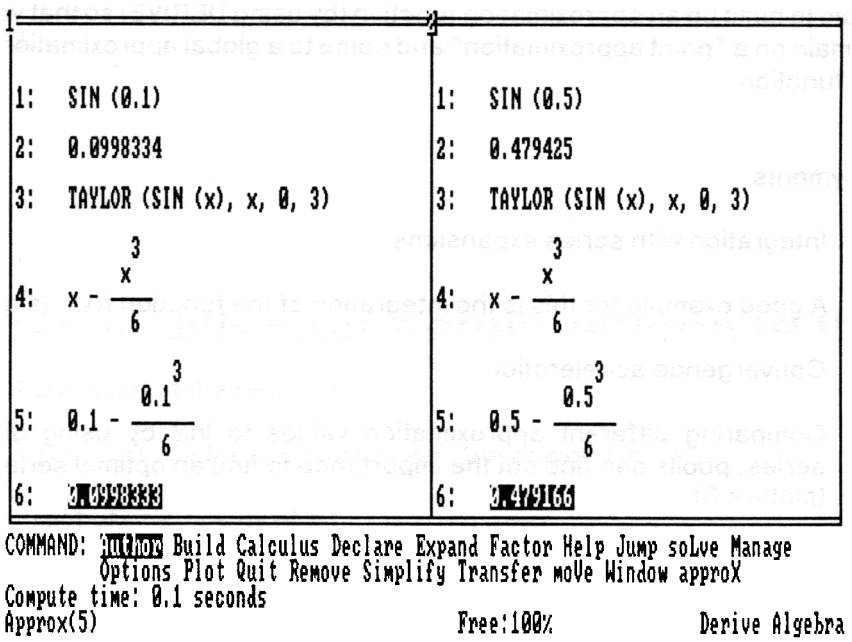
out of the expressions to the sine-functions plus to those of the cosine-function!"
(remember the equation of Euler: $e^{ix} = \cos x + i \cdot \sin x$)

The work with these instructions took quite a long time (about 60 minutes), but the pupils worked very intensely and concentratedly. And every pupil had his own success experience; everyone found out an answer to question 1, half of the class realize the factorials in agreement to the power of x and five of the 17 pupils found out the whole rule!

3. The following lessons

3.1 Computation of special approximation values

To show the pupils the importance of the nearness to the expansion point also from the algebraic side, you can compare the approximation values to sin0,1 and to sin0,5 given by the Taylor expansions in point 0. You get then (picture 2):



picture 2

By looking at and comparing these values, you can once more discuss, how important the nearness to the expansion point is for a good approximation. Here you can remember the graphics.

3.2 Deduction of the formula by using well-known themes

The equation $f(x) = T_n(x) + R_n(x)$ leads for n=0 to

$$R_0(x) = f(x) - f(0).$$

This expression is transformed by using the main formula of the differentiation and integration and the partial integration, so that you finally come to the general formula.

Proof

By using the ideas of the deductions the formula is proven with induction.

Generalization for a ε R

While formulating the general rule of Taylor it is very important to make clear two aspects to the pupils:
Firstly you must show them on special examples (using DERIVE), that only then a function can be expanded in the Taylor series when the remainder converges to 0. An example for a function, where it is not possible to expand into the Taylor series is the function

$f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$.

Secondly you must show to the pupils, that the Taylor series gives you firstly an approximation in the surrounding of one point. In better courses, it will be possible to build up an approximation function (by using DERIVE) so that you may not remain on a "point approximation" and come to a global approximation of the whole function.

Employments

3.5.1 Integration with series expansions

A good example for this is the integration of the function $f(x)= (\sin t)/t$.

3.5.2 Convergence acceleration

Comparing different approximation values to ln2 by using different series, pupils can find out the importance to find an optimal series. (picture 3)

Tabelle: "Näherungswerte zu ln2"

$\ln(1+x)$	$\ln\left(\frac{1+x}{1-x}\right)$
$\ln 2 = 0{,}693147..$	
$T_3(1) = 0{,}783333..$	$T_3(\frac{1}{3}) = 0{,}693004..$
$T_4(1) = 0{,}745634..$	$T_4(\frac{1}{3}) = 0{,}693146..$
$T_5(1) = 0{,}736544..$	$T_5(\frac{1}{3}) = 0{,}693147..$
$T_6(1) = 0{,}686055..$	
$\ln 3 = 1{,}09861..$	
	$T_3(\frac{1}{2}) = 1{,}09581..$
	$T_4(\frac{1}{2}) = 1{,}09849..$
	$T_5(\frac{1}{2}) = 1{,}09858..$
	$T_6(\frac{1}{2}) = 1{,}09861..$

1: LN (1 + x)

2: TAYLOR (LN (1 + x), x, 0, 4)

3: $-\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$

4: $\mathrm{LN}\left[\frac{1+x}{1-x}\right]$

5: $\mathrm{TAYLOR}\left[\mathrm{LN}\left[\frac{1+x}{1-x}\right], x, 0, 7\right]$

6: $\frac{2x^7}{7} + \frac{2x^5}{5} + \frac{2x^3}{3} + 2x$

7: $G(x) := -\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$

8: $H(x) := \frac{2x^7}{7} + \frac{2x^5}{5} + \frac{2x^3}{3} + 2x$

9: LN (2)

10: 0.693147

11: G (1)

12: 0.583333

13: $H\left[\frac{1}{3}\right]$

14: 0.693134

15: LN (3)

16: 1.09861

17: $H\left[\frac{1}{2}\right]$

18: 1.09806

picture 3

4. Possibilities for continuations to this unit

Estimation of the remainder

Taylor formula for polynomials with the power n (n ε N)

Proof of irrationality of e

Investigation of complex application problems (where the "realistic" functions are approximate by Taylor)

Finally I want to tell you, that this unit was much fun – not only for me, but also for the pupils. Between the first two lessons they didn't hear the school bell, because they were working so concentratedly. And they kept a lot in mind. At the end of the lessons (after three weeks vacations!!) we wrote a test and the results showed me, that even the "non-highflighers" got the most important points and formulas in their mind.