BACK

Using Computer Algebra to Teach the Foundations of Calculus

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A description is given of an approach to teaching the concept of limit and the foundations of calculus using the computer algebra package *DERIVE*. This involves the student carrying out a set of mathematical experiments, closely guided by a set of worksheets and based on some user-defined functions loaded into *DERIVE*.

Introduction

The Mathematics Department at City University has been using Computer Aided Learning methods for about six years. In the last two years, we have been using the computer algebra system *DERIVE*. Like most computer algebra systems, the package is fast and accurate, but, used on its own, we do not regard it as a teaching tool. Inevitably, there is a temptation to continue to teach mathematics in the traditional way and just to use the computer for the hard manipulative work. The computer and algebra package are used in much the same way as a calculator is used for arithmetic. The result is that the students learn no more mathematics than in the past and do not even acquire the manipulative skills they would previously have developed. To give an example, one could teach the basics of differentiation, show the student how to carry it out in *DERIVE* and send them away to evaluate a large number of examples using a computer. This would be a very poor way of teaching. At least in the past, the students learned some manipulative skills, even if they were very unsure of the theoretical basis of the operations they were carrying out.

The advantage of a computer algebra system is its ability to form the platform for the students to carry out mathematical experiments in a laboratory situation, in much the same way that a physicist or engineer carries out laboratory experiments. Of course, once the student has a good understanding of the fundamentals of a topic, there is no reason to discourage the use of the computer to take the hard work out of problem solving. It is hoped that their knowledge of the concepts and theoretical basis of a subject will be much better than in the past, since much less time need be spent in teaching routine manipulation.

There are mathematicians who feel that the loss of the ability to manipulate complex problems is a backward step and that we are not producing 'proper' mathematicians. If this were the only result, then they would have a case. However, we must take a positive approach and use the availability of the new tools to enhance our teaching and produce students with a much deeper understanding of the fundamentals of mathematics. Computer algebra systems exist, even if some wish they did not and we must use them to our advantage.

Mathematical experiments

The idea of numerical experiments of the 'what if' variety is familiar to any user of a spreadsheet. The same approach can be used to teach mathematics with a graph plotting or a symbolic algebra package. *DERIVE* combines a good symbolic algebra package with reasonable graphics in one software package. As an example, suppose we had declared a function of x, we could then examine its behaviour near a limiting value, without the need for any heavy calculations, just a few key strokes. I will discuss the implementation of this approach later.

There are two important elements in a laboratory situation. First, in the initial stages there should be close supervision, so that the students are never left staring at the screen not knowing what to do next. Once they have gained familiarity with the package, this close 'on-hand' help is not so important and the work is much more self-motivating. The second element is the provision of a carefully written worksheet for the 'experiments' being carried out. The worksheet should take the form of an easily read text, which guides the reader through the work in a stage by stage fashion. In addition, it should point out the important mathematical points, as they are seen on the screen, and try to make the students think about what they are doing in a mathematical way. The early worksheets in a set should give the commands that need to be given to the computer, even though this will take up a lot of space. Later sheets need be less specific in this area, since the students should be more confident in using the computer, leaving more space for exercises and explanation.

DERIVE has an extensive range of built-in functions, for example the commands ***C***alculus, ***D***ifferentiate quickly produces the derivative of a function. In the context of a teaching laboratory this may not give the desired *educational* result. I prefer to give my students sets of custom functions to enable them to carry out their experiments. These are loaded into the program as *DERIVE* utility files. The worksheet for the session carries a detailed description of the behaviour and purpose of the new functions, together with the syntax required for their use. More experienced students can be given the definitions of the functions, which they input at the keyboard. This approach encourages students to produce their own functions, when they require them.

The work described in this article uses the first approach, since it is aimed at the less experienced user.

There remains the question of 'thinking time'. In the early days of microcomputers, many of us wrote custom teaching programs for the machines then available. It was common practice to build time wasting loops into our programs to make them run slower. This was done to give the students time to think ahead and consider what the answer was going to be. It is possible to achieve this result with *DERIVE* by careful construction of the custom functions. Of course, once students are past the initial stages they should be shown the fast ways of obtaining their results.

Teaching the foundations of calculus

Perhaps the most difficult conceptual jump that students of mathematics have to make is when they first confront the concept of limit, usually when they begin their first course in calculus. In Britain, this is normally at age fifteen or sixteen. This concept is a quantum leap from anything they have met before, even the idea of rigorous proof can be introduced in easy stages and at an earlier age. Teaching this concept has always caused problems and frustration to both students and teachers. Anyone who has ever taught limit will probably have spent some time trying to convince an puzzled student that, for example, $\lim_{x \to 1} \frac{x^2+x-2}{x-1} \neq 1$. (For those who have not, the student's argument is as follows as $x \to 1$, $\frac{x^2+x-2}{x-1} \to \frac{0}{0} \to 1$).

It is now common for the first year at British universities in subjects such as accountancy, economics and the social sciences, to contain a course in calculus. Thus the problems in teaching the basics of calculus are no longer confined to the schools and technical colleges. The students on these courses have often stopped studying mathematics at the age of sixteen and so they have had a two year gap to forget the discipline of mathematics.

Until they meet the idea of limit, students have come to expect mathematics to be a subject of fixed, definite objects. Limit seems to contradict this outlook. In an attempt to solve this problem, I have produced a teaching package which introduces the ideas of limit and differential calculus. This is based on *DERIVE* and consists of a set of laboratory experiments to be integrated with normal classroom teaching. The intention is to make the transition to higher mathematics as painless as possible. The package is equally applicable to sixteen year olds in a school environment as to older students in a college or university.

A teaching package for limits and the foundations of calculus

The teaching material consists of two *DERIVE* utility files, together with a set of worksheets written to guide the student through some mathematical experiments. A utility file is a means of adding user defined functions to those built into *DERIVE*; details of these two files are given in the appendix. The purpose of the experiments is to introduce the concepts of the subject to the students at the same time as they are being taught them the classroom. It is hoped that this will lead to a deeper understanding of the mathematical concepts.

Limits

The theory of limits is taught by the first two worksheets in the set. The first sheet uses a set of numerical experiments to give some justification to the idea of limit, the second takes the students into a more symbolic treatment. The numerical experiments are based on the first utility file, *lim.mth* and consist first of defining a function, F(x), using the ***D***eclare, ***F***unction commands. The experimenter then looks at the possibility of a limit of the function at some point for which it is undefined. This is carried out with the user defined functions U(a,h) and L(a,h), which return the sequences of values {F(a+10h), F(a+9h), ..., F(a+h)} and {F(a-10h), F(a-9h), ..., F(a-h)}, respectively. Appro***X***imating these functions gives evidence that the function may have a limit at the point. For example, if $F(x) = (x^2+x-2)/(x-1)$ then U(1,.1) gives the sequence {4, 3.9, 3.8, 3.7, 3.6, 3.5, 3.4, 3.3, 3.2, 3.1}. The two functions also introduce the idea of upper and lower limit, although these terms are only mentioned in passing and are not discussed at length.

Once some experience has been gained with these functions and some limits found numerically, two new functions are introduced. These are U_LAST(a,h) and L_LAST(a,h), which produce the last term in the sequences, that is F(a+h) and F(a-h) respectively. This is not done by evaluating these functions, but by the rather convoluted means of generating the above sequence and picking off the last term. The purpose is to deliberately slow down the process and give the student 'thinking time'.

As in all the worksheets in this set, a set of examples is included for the user to try out the concepts they have learned. Seeing a function tending to a limiting *number* is a quick and easy way to become convinced of the underlying ideas of limit. It is no substitute for studying the abstract mathematics in a classroom, but it makes the concepts believable when they are introduced.

The second sheet on limit claims to introduce two new functions U_LAST(a) and L_LAST(a), for a numerical value of a these produce a function of h. An experienced user of *DERIVE* will know that this can be achieved with the earlier functions U_LAST(a,h) and L_LAST(a,h), but it was thought best not to explain this fact to the student. By ***D***eclaring ***F***unctions and ***S***implifying (not appro***X***imating) U_LAST or L_LAST for an appropriate value of a, a rather more rigorous idea of the limiting process is given, since the students have to consider what happens when $h \to 0$ for themselves. As an example consider the function $F(x):=\frac{x^2-1}{x-1}$ If we declare the function F(x) to be $(x^2-1)/(x-1)$ and then ***A***uthor and ***S***implify ***U_LAST(1)*** and ***L_LAST(1)*** we obtain ***2+h*** and ***2-h***, clearly these tend to 2 as h tends to zero. When using this method the students may need to do a little work themselves, the computer just takes away the worst of the drudgery. For example, when finding $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ it is necessary to multiply the numerator and denominator of the

result by $(\sqrt{1+h}+\sqrt{1-h})$. This is best done on paper. It is only at the end of the sheet that *DERIVE*'s own limit function is introduced, since by this point the students should have sufficient understanding of the concepts involved. To test their understanding they are given a set of examples to try by this direct method.

2. Calculus

The introduction to Calculus is restricted to Differential Calculus and consists of four worksheets. The first introduces the idea of a derivative by considering the limiting values of the slope of a chord to the curve of a given function. The next two sheets concentrate on developing techniques, such as the chain, product and quotient rule and to increasing the range of functions that can be differentiated. This work is concluded by the fourth sheet, which gives a practical and applicable point to the theory, the idea of maxima and minima.

The idea of differentiation is introduced through numerical experiments based on a group of functions defined in the second utility file, *diff.mth*. These are based on the function SLOPE(a,h) which gives the value of the slope of the chord between the points (a, F(a)) and (a+h,F(a+h)) for the previously declared function F(x). This enables us to define SLOPE(a) that gives the slope of the chord at x^2 as a function of h and SLOPES(a) that gives the set of values of SLOPE(a,h) for h = 0.1, 0.01, 0.001, 0.0001.

The students are asked to ***D***eclare ***F(x)*** to be the function x^2 and then to appro***X***imate the value of the function ***SLOPE(a,h)*** at various points x=a and for various values of h. It is intended that they should be able to see, 'experimentally', that its slope is given by 2x. They are then asked to repeat the exercise for x^3, x^4, x^5 and x^8, with the hope that they will 'discover' that the derivative of x^n, for a positive integer n, is nx^{n-1}.

Next, the experiments are made more abstract by repeating the above exercises, ***D***eclaring each of the functions and ***S***implifying ***SLOPE(a)***, where a is not replaced by a number. They are asked to look carefully at each solution and consider what happens when h is small. This should confirm to them that the derivative is nx^{n-1}. The rule is extended to negative integers and to rationals by asking them to look at x^{-5} and $x^{-½}$. The worksheet includes a set of examples to be done by hand. These give practice in the rule just learnt and are sufficiently easy so as not to require the computer's assistance.

The second sheet continues the use of SLOPE(a) with functions like x^3+x^2 so as to give an experimental justification to the sum rule of differentiation. It then defines a new function DERIVATIVE(1) that gives the first derivative of the previously declared function F(x). This function is used in a set of experiments to introduce the chain rule of differentiation. For example, the students are asked to ***D***eclare the ***F***unction ***F*** to be $(x+3)^3$ to ***S***implify ***DERIVATIVE(1)*** and to look for a pattern. It is felt that weaker students will require considerable help here, since the rule can only be 'found' by careful consideration of the results obtained.

At this point it was felt possible to introduce *DERIVE*'s normal method of finding derivatives and the students are taken through a set of trigonometric functions to increase their repertoire of derivatives and to amplify what they have learnt about the chain rule. Finally, the idea of higher derives is introduced and these are found using DERIVATIVE(2), DERIVATIVE(3), etc. It was felt that this was an appropriate place to introduce higher derivatives, since the concept is not difficult and the function DERIVATIVE(n) provides an easy method of finding them.

The third calculus sheet is devoted to the product and quotient rules. The utility file contains functions U and V, declared to be functions of x together with PROD(x) and QUOT(x) which are the product and quotient of U and V respectively. The student are asked to ***D***eclare the functions ***U*** and ***V***, then to author ***PROD(x)*** and use the commands ***C***alculus, ***D***ifferentiate to find the derivative. The experimental part is for them to try to spot the product rule from the result. A set of examples is given which should help in finding the rule and to confirm it when it is found. A further function loaded with *diff.mth* is *prod_rule*. This is defined in the worksheet and is just the product rule in *DERIVE*'s notation. The students are encouraged to try it for various U and V, first to reinforce the product rule and second to give them sight of a user defined function.

A similar approach is used for the quotient rule, requiring them to differentiate the function QUOT(x) using *DERIVE*'s ***C***alculus, ***D***ifferentiate commands. A function *quot_rule* is also introduced in the same manner as for products.

The experimental approach used here gives no theoretical justification for these rules, but such experiments give a feeling of discovery to the learning process. If the theory is taught in the classroom at about the same time then the two approaches should be mutually supportive.

The final worksheet in the set teaches the basics of turning points, maxima and minima. Unlike the other sheets, it makes use of the graphics facilities of *DERIVE*. This is done with the screen split into two windows, an algebra window and a two dimensional plot window. The students are first asked to plot the simple function $2x-x^2$, to observe its turning point and to find its value by placing the cross-wires on it. Then they are asked to find the derivative of the function and to find the values of x for which it is zero. Of course, this turns out to be the point at which the maximum occurs and they are asked to consider why this happens. Points of inflexion are introduced by giving the example x^3. No attempt is made to give the theoretical basis of points of inflexion, but an appeal is made to common-sense by talking about points where the curve 'stops bending'.

This worksheet makes no attempt to give the various tests which enable us to classify the turning points found. Instead, the student is asked to plot the functions and classify them from the shape of the curve. It was felt that this caused less confusion and, of course, this is really a visual version of the first derivative test.

A new function TURN(y) is introduced that only has to be simplified to find the turning points of y. For example, ***S***implifying ***TURN(x^3-x)*** gives $\pm\sqrt{3}/3$. The students are asked to plot the graph of various functions and by simplifying TURN,

to find and classify their turning points. The relatively simple function, TURN, which is defined on the worksheet, demonstrates how a user defined function can save a lot of keystrokes. It is hoped that this will encourage the students to define their own functions when they need them.

It was decided that this was an appropriate point to finish these initial experiments in calculus. The basic principles of differential calculus have been taught and the last sheet has given a practical motivation to the idea of differentiation.

Summary

The main benefit that the laboratory sessions can bring to the teaching process is the generation of enthusiasm among the students. This is mainly achieved through the feeling of discovery that the experiments give them. It is hoped that this will stimulate their interest when the theory is taught, away from the computer, and that they will be encouraged to ask searching questions. For this reason, these experiments must be fully integrated with the normal classroom work: they are of limited value in their own right.

Appendix

This is a listing and explanation of the two *DERIVE* utility files used in the teaching package described in the article.

1. Lim.mth

This file is used in the first two worksheets teaching the concept of limit.

Declare F to be an unspecified function of x.

F(x):

Generate a sequence on F(x) from x = a + 10h to x = a + h, to enable us to see if there is a possibility of an upper limit.

U(a,h): = VECTOR(F(x),x,a + 10*h,a + h,-h)

Look at last term in sequence for upper limit, we could of course define this to be F(a + h), but this was felt to simplify too quickly, the present method is slower, giving 'thinking time'.

U_LAST(a,h): = ELEMENT(U(a,h),10)

Generate a sequence on F(x) from x = a-10h to x = a-h, to enable us to see if there is a possibility of a lower limit.

L(a,h): = VECTOR(F(x),x,a-10*h,a-h,h)

Look at last term in sequence for lower limit, again this could be achieved with F(a-h), but this gives little 'thinking time'.

L_LAST(a,h): = ELEMENT(L(a,h),10)

2. diff.mth

This file is used for the for worksheets used to teach the foundations of calculus.

Declare F to be an unspecified function of x.

F(x): =

Find slope of chord from x a to x a+h, the choice of a and a+h rather than a-h and a+h is deliberate.

SLOPE(a,h): = (F(a+h)-F(a))/h

Vector showing the value of the slope for h = 0.1, 0.01, 0.001, 0.0001, to enable investigation of the possibility of a limit.

SLOPES(a): = VECTOR(SLOPE(a,10^(-k)),k,4)

A function which returns the n-the derivative of F(x).

DERIVATIVE(n): = DIF(F(x),x,n)

The following functions are used for experiments with the product & quotient rules and turning points.

Declare U and V to be unspecified functions of x.

U(x): =

V(x): =

Define product and quotient functions

PROD(x): = U(x)*V(x)

QUOT(x): = U(x)/V(x)

Define functions to test the product and quotient rules

prod_rule: = DIF(U(x),x)*V(x) + DIF(V(x),x)*U(x)

quot_rule: = (DIF(U(x),x)*V(x)-U(x)*DIF(V(x),x))/V(x)^2

The following function is used to find turning points.

Define a function to return the turning points of the specified y.

TURN(y): SOLVE(DIF(y,x),x)

Limits and Calculus

Limits 1

This worksheet introduces you to the idea of the 'limit' of a function. A limit is the value that a function approaches when the value of the dependent variable gets very close to some number for which the function itself is undefined. For example, the function $\frac{x^2-1}{x-1}$ gets closer and closer to the value 2 as x gets close to 1, ***even though the function itself is undefined for x = 1.*** You can easily see this using Derive First ***A***uthor the function $\frac{x^2-1}{x-1}$, then, **keeping it highlighted**, use the commands ***M***anage, ***S***ubstitute to give different values to x and then use appro***X***imate to find the value of the function. Try this for x = 1.1, 1.01, 1.001, 0.9, 0.99, 0.999. Hopefully, you should now be convinced that the function really does have a 'limit as x tends to 1', (written x→1), namely 2.

The above procedure is rather time consuming, you could do it almost as fast on your calculator. We need to automate the process to enable us to look at the possible limits of lots of functions. First, we set Derive to show 12 figures in any approximate calculations we may make. This is done with the commands: ***O***ption, ***P***recision, ***<Tab>***, ***12***. Then we must load some extra functions into Derive using the following commands: ***T***ransfer, ***L***oad, ***U***tility and when asked for a filename answer ***lim.mth***. The new functions are listed below, with an explanation of what they do:

F(x) This is the function whose limit we are considering, for example in the above it would be $\frac{x^2-1}{x-1}$ Before you can look at any limits you must, of course, first define your function. This is done with the commands ***D***eclare, ***F***unction, ***F***, ***<Enter>***, ***(x^2-1)/(x-1)***. Of course, this gives the above function, you can input whatever you wish as your function. **For the moment put in (x^2-)/(x-1).**

U(a,h) This gives the set of values of F(x) for x = a + 10h, a + 9h, ..., a + h, enabling us to see if the function seems to be tending to a limit. For example, in the above, we could ***A***uthor, ***U(1,.1)*** and then appro***X***imate. We could then repeat this with ***U(1,.01)***. **Try it and see what values you get.**

U_LAST(a,h) This function is a useful tool that just gives the last value in the set of numbers given by U(a,h). If you use U(a,h) you will need to track along the numbers with the cursor keys, but after you have spotted a trend, this new function will enable you to quickly look at the most interesting term in the set, the last one. Try ***A***uthoring and appro***X***imating ***U_LAST(1,.001)*** and ***U_LAST(1,.00001)*** does this begin to confirm your suspicion that there is a limit of 2.

So far we have always approached the point from above, x=1.1, 1.01, etc. Suppose we came in from below, x=0.9, 0.99, .., would we get the same limit? If **not** then we can't talk about **THE** limit as x→1, we can only do so if the same result is obtained whatever way we come in. We need some new functions to check for the limit from below, they are as follows:

L(a,h) This gives the set of values of F(x) for x=a-10h,a-9h, ...,a-h, enabling us to see if the function seems to be tending to a limit. Try ***A***uthoring and appro***X***imate the function ***L(1,.1)*** and then perhaps ***L(1,.01)***.

L_LAST(a,h) This function is a useful tool that just gives the last value in the set of numbers given by L(a,h). If you use L(a,h) you will need to track along the numbers with the cursor keys, but after you have spotted a trend, this new function will enable you to quickly look at the most interesting term in the set. Try ***A***uthoring and appro***X***imating ***L_LAST(1,.001)*** and ***L_LAST(1,.00001)*** does this confirm your suspicion that the limit is 2.

Strictly, ***U(1,-.1)*** and ***L(1,.1)*** are the same thing, but we will keep to positive values of h to save confusion. Now we will apply our new functions to test for the possibility of some limits. It is important to realise that, for the moment, we are only carrying out numerical experiments, we are not looking at any abstract mathematics.

The notation used for a limit is $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$. (i.e.limit of $\frac{x^2-1}{x-1}$ as x tends to 1 is 2). This is only meaningful if both the upper limit, given by U(a,h) and the lower limit, given by L(a,h) are equal. If they are NOT equal, we are unable to talk about **the limit**.

Use your new functions to find, experimentally, the following limits, *if they exist*, in each case check that the limit given by U_LAST(a,h) and L_LAST(a,h) are the same.

Important: For each of the following, you should use ***D***eclare, ***F***unction, ***F*** and then type in the function definition, DO NOT USE ***A***uthor to define F(x), but use ***A***uthor for L_LAST(a,h) and U_LAST(a,h).

1 $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$

2. $\lim\limits_{x\to 0}\dfrac{x+\frac{1}{x}}{x-\frac{1}{x}}$

3. $\lim\limits_{x\to -2}\dfrac{x^3+4x^2+x-6}{x^2+6x+8}$

4. $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{x^2-4x+4}$

5. $\lim\limits_{x\to -1}\dfrac{x^4-2x-3}{x+1}$

6. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

7 $\lim\limits_{x\to 0}\dfrac{\sqrt{5+x}-\sqrt{5}}{2x}$

8. $\lim\limits_{x\to -3}\dfrac{x+3}{\sqrt{x^2+7}-4}$

9. $\lim\limits_{x\to -1}\dfrac{\sqrt{x^2+3}-2}{x+1}$

10. $\lim\limits_{x\to 3}\dfrac{\sqrt{2x+3}-x}{x-3}$, $x\geq -\dfrac{3}{2}$

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Limits and Calculus

Differentiation 1

This worksheet introduces you to the basic concepts of differential calculus. We begin by considering the slope of the chord to the curve $y = F(x)$, between the points $(a, F(a))$ and $(a+h, F(a+h))$ and then we experiment to see how the value of the slope of this line changes as h gets very small. In the limit we are looking at the slope of the tangent to the curve at the point $(a, F(a))$.

First you should load some extra functions into Derive using the commands: ***T***ransfer, ***L***oad, ***U***tility and when asked for a filename answer ***diff.mth***. The new functions are listed below, with an explanation of what they do:

F(x)	This is the function whose slope we are considering. For example, if we wish to look at $F(x) = x^2$, we would ***D***eclare, ***F***unction, ***F***, ***<Enter>***, ***x^2***, ***<Enter>***.
SLOPE(a,h)	This gives the value of the slope of the chord to the curve between the points $(a, F(a))$ to $(a+h, F(a+h))$. If you do not give numeric values to a and h, you will obtain a symbolic formula for the slope.
SLOPE(a)	This gives the slope of the chord to the curve between the points $(a, F(a))$ and $(a+h, F(a+h))$ as a function of h. If you do not give a numeric value to a, you will obtain a formula for the slope, in terms of a and h.
SLOPES(a)	This gives the set of values of SLOPE(a,h) for $h = 0.1$, 0.01, 0.001 and 0.0001.

First, we ***D***eclare the ***F***unction ***F*** to be ***x^2*** and then ***A***uthor and appro***X***imate the following: ***SLOPE(2,.1)***, ***SLOPE(2,.01)***, ***SLOPE(2,.001)***, ***SLOPE(2,.0001)***. Can you see a possible limiting value? Now, look at ***SLOPES(3)*** and ***SLOPES(4)***, can you find a possible limiting value for these? In fact, the slope of $y = x^2$ is given by the formula 2x, does this fit the results you have obtained?

Now ***D***eclare the ***F***unction ***F*** to be ***x^3*** and repeat the process. Can you guess the formula for slope of this curve. Try again for $\mathbf{y = x^4}$, $\mathbf{y = x^5}$ and $\mathbf{y = x^8}$ can you find the formula for $y = x^n$, where n is an integer and $n > 0$?

We now approach the problem from a in a more mathematical way, using the symbolic power of Derive. ***D***eclare the ***F***unction ***F*** to be ***x^2*** again and then ***A***uthor and ***S***implify ***SLOPE(a)***. *Do not replace a by a number.* Consider the value of the resulting formula when h is very small, does this agree with your earlier result? Try this for ***x^3***, ***x^4***, ***x^5*** and ***x^8***, does your guess for the formula for x^n seem to be correct?

The limit of the slope of the chord is the slope of the tangent at x = a and is called the **Derivative of F(x) at x = a**, it is written as $\frac{dF}{dx}$. We now extend our research to see if the formula we suspect, namely $\frac{d(x^n)}{dx}=nx^{n-1}$, holds for integer values of n such that n < 0, or for rational values of n such as -½

Declare the ***F***unction ***F*** to be ***x^-5*** and ***A***uthor and ***S***implify ***SLOPE(a)***, try again for **x^(-1/2)**, in this last case, you will need to multiply the numerator and denominator by $\sqrt{a}+\sqrt{a+h}$, *by hand.*

Use what you have discovered to find the derivative of the following functions, **without using the computer**:

1 x^{10} 2. $4x^7$ 3. $2x^{-8}$ 4. $4x^{-\frac{3}{2}}$

5. $\frac{1}{x^4}$ 6. $\frac{1}{\sqrt{x}}$ 7. $({}_3\sqrt{3x})^2$ 8. $\frac{7}{({}_7\sqrt{x})^4}$

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Limits and Calculus

Differentiation 2

This worksheet guides you through some more mathematical experiments to increase your knowledge of differentiation.

First we must load some extra functions into Derive using the commands: ***T***ransfer, ***L***oad, ***U***tility and when asked for a filename answer ***diff.mth***. The functions we will use are:

F(x) This is the function we are differentiating. For example, if we wish to look at $F(x)=x^2$, we would ***D***eclare, ***F***unction, ***F***, ***<Enter>***, ***x^2***, ***<Enter>***.

SLOPE(a) This gives the slope of the chord to the curve between the points (a,F(a)) and (a+h,F(a+h)) as a function of h. If you do not give a numeric value to a, you will obtain a formula for the slope, in terms of a and h.

DERIVATIVE(1) This gives the derivative of F(x), note the **1**, for the moment don't try any other numbers.

In the last sheet we saw that the derivative of x^n was nx^{n-1}. What happens if we wish to differentiate the sum of two such powers, x^n and x^m, say, is the result the sum of the two derivatives, that is $nx^{n-1}+mx^{m-1}$? It would be unwise to take this on trust, so try the following experiment: ***D***eclare, ***F***unction, ***F***, ***<Enter>***, ***x^3+x^2***, ***<Enter>***, N.B. do not use ***A***uthor. Then ***A***uthor, ***SLOPE(a)*** and ***S***implify, look at the result you get and see if, when h is small, you are left with $3a^2+2a$. This seems to confirm our hypothesis, but perhaps we should look for a little more evidence. Use the commands: ***D***eclare, ***F***unction, ***F***, with the following functions to see if SLOPE(x) gives the expected result.

1 x^3+3x-4 2. $x^8+2x^6-4x^3+6x+9$

In the next example,it is easier to use the commands ***M***anage, ***S***ubstitute to set h as zero, rather than try to look for the limit by hand, (be careful of this approach, it will not always yield the desired result).

3. $2x^4-3x-2x^{-4}$

Now, we can look at the new function **DERIVATIVE(1)**, this gives the derivative of F(x), ***A***uthor and ***S***implify ***DERIVATIVE(1)*** to check your last result.

Suppose we wish to differentiate a composite function like $(x+3)^3$, is there a rule to enable us to find the result easily? ***D***eclare the ***F***unction ***F*** to be $(x+3)^3$, and then ***A***uthor and ***S***implify ***DERIVATIVE(1)***, can you see a pattern? Try the following and again try to guess the rule, (known as the 'chain rule').

1 $(x-3)^7$ 2 $(x+3)^{-3}$ 3 $(4x-7)^5$

4. $(x^2+1)^3$ 5. $(2x^2+3x-1)^4$ 6. $\dfrac{2}{(3x+5)^6}$

7. $(x^2+4x-3)^5$ 8. $(x^2+3x-2)^{-5}$ 9. $\dfrac{1}{(x^2+3x+2)}$

Derive has its own, built-in, method of finding derivatives, you should highlight the function and then give the following commands: ***C***alculus, ***D***ifferentiate, ***<Enter>***, ***<Enter>***, ***1***, ***<Enter>*** and then ***S***implify the result. Try it on some of the above examples and then extend your knowledge by finding the derivative of the following:

1 $\sin(x)$ 2. $\cos(x)$ 3. $\tan(x)$

4. $\sin(3x^2+2x-1)$ 5. $\cos^2(2x-1)$ 6. $\dfrac{1}{sin2x}$

(You may ***A***uthor, these functions rather than use ***D***eclare ***F***unction).

You may have wondered the purpose of the 1 in DERIVATIVE(1), or the question *order* in Derive's command sequence to find a derivative. Of course, you can differentiate a derivative, for example x^4 has *first derivative* $4x^3$ and *second derivative* $12x^2$. DERIVATIVE(2) gives the second derivative of F(x) and DERIVATIVE(3) the third etc. Use ***D***eclare, ***F***unction, ***F*** and ***DERIVATIVE*** to find the first, second and third derivatives of the following, check your results using Derive's ***C***alculus, ***D***ifferentiate:

1 $3x^7$ 2. $(5x-4)^6$ 3. $\dfrac{1}{(x-3)^3}$

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Limits and Calculus

Differentiation 3

This worksheet guides you through some experiments to discover how to find the derivatives of functions made of the product of two functions, like $(X^2-3x+5)^3(x-1)^2$, or a quotient like $\dfrac{x^3-3x^2+5}{(x^2+2)^3}$.

First we must load some extra functions into Derive using the commands: ***T***ransfer, ***L***oad, ***U***tility and when asked for a filename answer ***diff.mth***. The functions we will use are:

U(x), V(x) These are the functions we will be differentiating, they must be declared with ***D***eclare, ***F***unction in the normal way.

PROD(x) This gives the product U(x)V(x)

QUOT(x) This gives the quotient U(x)/V(x).

To begin our investigation use ***D***eclare, ***F***unction, ***U***, ***Cos(x)*** and ***D***eclare, ***F***unction, ***V***, ***(x² + 1)***, then ***A***uthor, ***PROD(x)***. Now, find the derivative of the product, $(x^2+1)\cos(x)$, by highlighting PROD(x) and giving the commands: ***C***alculus, ***D***ifferentiate, ***<Enter>***, ***1***, ***<Enter>***, ***S***implify. Can you see a 'rule' for finding the derivative of a product? If you can, see if it works for the following pairs of U(x) and V(x), (N.B. in each case you must ***D***eclare the ***F***unctions ***U*** and ***V***). If you *can't* see the pattern, try them and see if they help you spot the rule.

1 $x^2(x+1)^4$ 2 $x^3\cos(x)$ 3 $x^4(1+x^2)^2$

4. $(1+x)^{\frac{1}{3}}(2-x^2)^{\frac{3}{5}}$ 5. $(x+1)^3(1-3x^3)^5$ 6. $\cos(x^2)\sin(3x^2-2x)$

If you were unable to spot the rule it is as follows:

$$\frac{d}{dx}(UV) = \frac{dU}{dx}V + U\frac{dV}{dx}.$$

There is a function loaded with *diff.mth* which evaluates this rule, it is called ***prod_rule***. The formula is: *prod_rule:=DIF(U(x),x)*V(x)+DIF(V(x),x)*U(x)*. Try ***A***uthoring and ***S***implifying ***prod_rule*** for some of the above examples.

We now turn our attention to the quotient of two functions. Begin by again using ***D***eclare, ***F***unction, ***U***, ***Cos(x)*** and ***D***eclare, ***F***unction, ***V***, ***(x^2+1)***, then ***A***uthor, ***QUOT(x)***. Now highlight QUOT(x) and give the commands: ***C***alculus, ***D***ifferentiate, ***<Enter>***, ***<Enter>***, ***1***, ***<Enter>***, ***S***implify. Again, can you see a 'rule', its not so easy this time, try writing the result over a common denominator. Attempt the following examples and see if you can see a pattern (NB in each case you must ***D***eclare the ***F***unctions ***U*** and ***V***):

1 $\dfrac{x}{1-x^2}$ 2. $\dfrac{\cos(x)}{\sqrt{x}}$ 3. $\dfrac{1+\sin(x)}{1-\sin(x)}$

The quotient rule is in fact:

$$\frac{d}{dx}\left(\frac{U}{V}\right) = \frac{\frac{dU}{dx}V - U\frac{dV}{dx}}{V^2}$$

There is a function loaded with *diff.mth* which evaluates this, it is called ***quot_rule***. The formula is: *quot_rule:=(DIF(U(x),x)*V(x)-U(x)*DIF(V(x),x))/V(x)^2*. Try ***A***uthoring and ***S***implifying ***quot_rule*** for some of the above examples.

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Limits and Calculus

Turning Points, Maxima and Minima

In this worksheet we will be looking at some of the geometric applications of differentiation and its use in finding the local maxima and minima of functions.

To facilitate our geometric work, we begin by splitting the screen into two windows, one for algebra and one to draw graphs. We do this with the following commands: ***W***indow, ***S***plit, ***V***ertical, ***40***, ***<Enter>***, this splits the screen into two windows. You should note that the switch from one window to the other is ***<f1>***, *try it*, note that the number of the active window, in the top left-hand corner is back-light. We will now designate one of the windows as a 2-D plot window. Switch to the window you wish to make your plot window and give the commands: ***W***indow, ***D***esignate, ***2***-D, if you are asked the questions *'Abandon expressions Y/N?'*, answer **Y**, you have two copies of the expressions! You should now have a screen split into two, as desired, with the ability to switch between them.

In your Algebra window ***A***uthor $2x-x^2$, switch to the plot window with ***<f1>***, and ***P***lot the function. You will see that the function has a turning point, where the graph reaches a 'local maximum', at a point with $x>0$. Move the cross-wires to the point with the cursor keys and you will find it is the point (1,1). We are interested in predicting such points, without needing to plot the curve.

Switch back to the Algebra window with ***<f1>*** and load the Utility file diff.mth with the commands: ***T***ransfer, ***L***oad, ***U***tility and when asked for a filename answer ***diff.mth***. The function we will use will be described later. For the moment, highlight the function $2x-x^2$ and use ***C***alculus, ***D***ifferentiate to find a formula for its derivative.

Let us consider for what values of x the derivative is zero, we find these by highlighting the formula for the derivative and using the commands: so***L***ve, ***<Enter>***. This gives the values of x for which the derivative is zero, you should note that this is just the value of x where the maximum occurs. If you look at the graph, this is reasonable, since at the maximum the curve is 'level' and thus the slope is zero. A little thought will convince you that this is also true for minima.

Try the following experiment to emphasis the point: ***A***uthor, $4x^2-4x+5$, then use ***C***alculus, ***D***ifferentiate to find a formula for the derivative. Now so***L***ve this to find the points where the slope is zero. Highlight $4x^2-4x+5$, switch to the Plot window and ***P***lot the curve, you will note that x = ½ gives a minima of the function. *If you can't see the part of the curve you want in the window, then it can be re-sized:* ***<f10>*** *zooms out - makes the scale smaller, whilst* ***<f9>*** *zooms in. Note also that you can clear the plot window with the* ***Delete*** *command.*

It would seem that all we need to do to find the maxima and minima for a function is to differentiate it, set the result equal to zero and solve the equation. We can then tell the points apart by plotting the graph. Clear the Plot window, switch to the Algebra window and ***A***uthor, x^3-x, then find the values of x which give the maxima and minima, ***P***lot the graph to confirm your experiment.

This method is not perfect, for instance try x^3, plotting should convince you that x = 0 is neither a maxima or minima, in fact it is called a point of inflexion. This is a point where the curve, momentarily, stops 'bending' and then continues in the same direction. However, the method does seem to yield all the maxima and minima. These points, together with the points of inflexion are called ***turning points*** for obvious reasons.

We can automate the process with the function **TURN(y)**, this is included in *diff.mth*. Its formula is TURN(y): =SOLVE(DIF(y,x),x) and it yields the values of x for which $\frac{dy}{dx}=0$. Try ***A***uthoring and ***S***implifying, ***TURN(x³-x)***, you will obtain the values $\pm\sqrt{3}/3$.

Use the function TURN to find the x values of the turning points of the following functions and classify the nature of the points, by plotting the curves in your plot window:

1 x^2-3x+4

2. $x^4-4x^3-2x^2+12x-8$

3. $x^2+\frac{1}{\ }$

4. $(x+2)(x-2)^3$

5 $\frac{4x}{\ }$

6. $x^2\sqrt{5+x}$

7. $x^{\frac{1}{3}}(x+2)^{-\frac{2}{3}}$

8. $x^2\sqrt{3-x^2}$

9. $\frac{x(x-1)}{(x+1)(x-5)}$

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Limits and Calculus

Limits 2

This worksheet carries on from the numerical experiments we carried out in Limits 1. However, this time we will work symbolically, rather than numerically. This will give us more insight into the nature of a limit.

First we must load in the new functions we used last time. We do this with the following commands: ***T***ransfer, ***L***oad, ***U***tility and when asked for a filename answer ***lim.mth***.

The file contains two extra functions which we will use in our experiments, they are:

U_LAST(a) This function is gives the last value in the set of numbers given by U(a,h). Unlike U_LAST(a,h), which produced a number, it produces, after ***S***implification, the value of F(a + h). You should then be able to spot the value of the limit, if one exists, by considering what happens if h is small.

L_LAST(a) This function is gives the last value in the set of numbers given by L(a,h). Unlike L_LAST(a,h), which produced a number, it produces, after ***S***implification, the value of F(a-h). You should then be able to spot the value of the limit, if one exists, by considering what happens if h is small.

As an example consider the function $F(x):=\frac{x^2-1}{x-1}$. If we use ***D***eclare, ***F***unction, ***F***, ***<Enter>***, ***(X^2-1)/(x-1)*** and then ***A***uthor and ***S***implify ***U_LAST(1)*** and ***L_LAST(1)*** we obtain ***2+h*** and ***2-h***, which clearly tends to 2 as h gets small. When using this approach you may need to do a little work yourself (***in your head!!***), the computer just takes away a lot of the drudgery.

We will look at the same examples we used in the previous sheet and compare our answers with those we got last time.

For each of the following, use ***D***eclare, ***F***unction, ***F*** to define the function (DO NOT USE ***A***uthor to define F(x)), then ***A***uthor and ***S***implify ***U_LAST(a)*** and ***L_LAST(a)*** for an appropriate value of a to find the following limits:

1 $\lim_{x\to 2} \frac{x^2-4}{x-2}$ 2. $\lim_{x\to 0} \frac{x+\frac{1}{x}}{x-\frac{1}{x}}$ 3. $\lim_{x\to -2} \frac{x^3+4x^2+x-6}{x^2+6x+8}$

4. $\lim_{x\to 2} \frac{x^2-x-2}{x^2-4x+4}$ 5 $\lim_{x\to -1} \frac{x^4-2x-3}{x+1}$

The next example this procedure leads to an indeterminate result, to obtain an answer multiply the numerator and denominator of your result by $(\sqrt{1+h}+\sqrt{1-h})$, using a piece of paper!

6. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Try a similar method for the next example

7. $\lim_{x\to 0} \frac{\sqrt{5+x}-\sqrt{5}}{2x}$

In fact, Derive is able to find limits with one of its built in functions. First, highlight the function $\frac{\sqrt{5+x}-\sqrt{5}}{2x}$ and then use ***C***alculus, ***L***imit, ***<Enter>***, ***<Enter>***, ***-2*** (in this case), ***<Enter>***, then ***S***implify for the result.

Use this method to find the following three limits, (*in each case you should **A**uthor the function and use **C**alculus, **L**imit).*

8. $\lim_{x\to -3} \frac{x+3}{\sqrt{x^2+7}-4}$ 9. $\lim_{x\to -1} \frac{\sqrt{x^2+3}-2}{x+1}$ 10. $\lim_{x\to 3} \frac{\sqrt{2x+3}-x}{x-3}$ $x \geq -\frac{3}{2}$

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