BACK

USING DERIVE TO TEACH MATHEMATICS FOR COMPUTER SCIENCE STUDENTS

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Abstract: The Department of Applied Mathematics of the Computer Science Technical College at the Universidad Politécnica of Madrid is introducing the use of computers in its teaching, developing tutorial and practical lessons for different topics of the mathematical subjects. An explanation is given of the main ideas of our methodology, as well as some examples of different practices and comments about the achievements of this experience.

1. INTRODUCTION

The teachers of the Department of Applied Mathematics to Computer Science at the Universidad Politécnica of Madrid (U.P.M.) consider that one of our responsibilities is to make known, as widely as possible, the new technology and tools used at present in the area we are teaching. What is more, as we are in a Computer Science College, to ignore the impact of technology on present and future practice of mathematics, engineering, science, etc. would be a fraud upon our students.

Hence, one of our Department's most important lines of work is to introduce the use of different computing tools in our teaching, as well as to encourage these kind of experiences in other Colleges and Spanish Universities. In that way, we organized in September 1990 a meeting on "Experimental Teaching in Mathematics", aimed to U.P.M. teachers, and another one on "Experimental Teaching in

Mathematics at University (Madrid, December 1991) on a nation-wide scale.

Among the different ways to use computers in mathematical instruction (drill and practice, tutorial systems, microworlds, writing programs, etc.) we are developing "show and tell" experiences (with large screen displays to make illustrations in an ordinary class) and, specially, tutorial and practical lessons in the Computer Center (we could call that a Mathematics Laboratory). Refering to the tools we are using, one of the most attractive and effective of them is the symbolic computation system Derive.

Our main objective is not for students to perfectly know the use of each computing tool, but to give them the opportunity of reflecting and working on the mathematical concepts in a different way. We can quote from [13]: "The language or the tool, in and of itself, is never enough. What makes the difference is the set of activities in which students engage, and the ways they make use of the language or tool. It is not what is on the screen that counts, it is what is in the student's head."

The main students' task won't be to compute quickly, but to analyse the results the computer shows them. Quoting from [15] "In the world our students will be inhabit, the succesful individual will live in the dictum: 'Machines compute, people think'.". In that way, to emphasize reflection and thinking we seize the pedagogical advantages of computers (computing speed, interaction, visualization, learning from errors, feedback, etc.) and some suitable properties of, for instance, Derive (ease of use, menu driven, graphical posibilities, symbolic computation, formal requirements, etc.).

Introducing the use of Derive as a thought–provoking tool has implied a creative effort to propose new suitable activities for its use. The problems posed in the traditional lessons were either too theoretical (Derive cannot solve them) or calculation exercises (Derive works them out by pressing "a couple of keys"). We wanted to give priority to the students' activity and creativeness.

Thus, we have designed a different set of problems related to some topics of the subjects Calculus, Algebra and Numerical Analysis, which the students, using Derive in tutorial and practical lessons, have to think about.

For each practice, they make use of a script, where the successive steps are outlined, and fill in an answer sheet (a series of questions with spaces for answers and comments). However, the important thing is not the final result, but the process: to observe, to reflect, to correct, to try new solutions, ... The result we pursue is the students' learning.

Below, an explanation will be given about some of the more interesting exercises.

2. EXAMPLES OF SOME EXERCISES WITH DERIVE

2.1 Identifying graphics.

One of the objectives of any course of Calculus in first year at University is for students to know the reciprocal relationship between the graphic of a function, its analytical expression and its analytical properties (continuity, derivability, limits, extrema, etc.).

Graphical and computing posibilities of Derive allow us to develop a new methodology to teach this topics. This methodology, as we have said above, emphasizes the process (observing, reflecting, correcting, ...) rather than the results.

Two kind of problems are posed:

- Given the analytical expression of some functions, students have to plot them using Derive, and, considering the graph, both infer the properties of the functions related to continuity, derivability, limits, extrema, etc., and generalize some properties for a specific class of functions (for instance: for polynomials, how the limit as x tends to infinity depends on their degree).
- Given the graph of a function, students have to answer some questions about its analytical properties. After that, and considering them, they have to find out, testing with Derive, an analytical expression which graph is similar to the given at first. When they have arrived at some suitable expression, they have to check its analytical properties, comparing them with the ones they observed before from the graph.

An example of the second kind of problem is given below.

1 Let f be the function which graph is in *Fig. 1.*

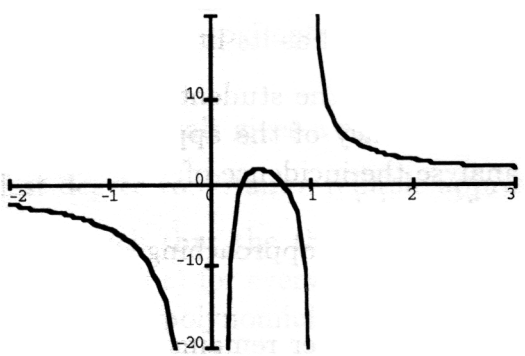


Fig. 1

1.1 Considering the graph, answer this questions:

a) Find $\lim_{x \to \infty} f(x)$.

b) Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$.

c) How many roots has f?

d) How many singular points has f?

e) Has any asymptote'

1.2 Testing with Derive, find out an analytical expression for a rational function whose graph is similar to the one in *Fig. 1.*

1.3 Call the function found as g and, using Derive

a) Find $\lim_{x \to \infty} g(x)$, $\lim_{x \to 0} g(x)$ and $\lim_{x \to 1} g(x$

b) Find the zeroes of g

c) Find the extrema of g

d) Find the asymptotes of g.

2.2 Taylor Polynomials

In every course of Calculus, Taylor's Theorem appears both as fundamental for local analysis of real functions and as a polynomial approach of the function's values.

In our context of Computer Science studies, the students' comprehension of both concepts and techniques related to approximation of functions is one of our most important objectives.

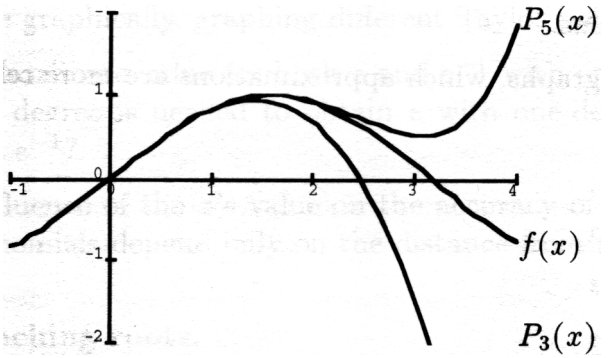
Hence, a practice about "Taylor Polynomials" has been developed seizing Derive's facilities to present some basic ideas about approximation to functions by polynomials (approximated values, errors, accuracy of an approximation, ...) and, of course, the specific properties of Taylor polynomial approximation (local character, incidence of the degree in the accuracy of the approximation, Lagrange's form of the remainder, ...) as well as its limitations (interval of convergence).

With each problem we want the students to become aware about how each parameter influences the accuracy of the approximation. In that way, they are successively asked to analyse the incidence of:

- the degree of the polynomial
- the point which value they are approaching
- the specific function.

In each problem, each parameter remains fixed, except the one that is analysed, and students are asked to compare the differences as that parameter varies.

To be more intuitive, most of the questions posed need observing and reflecting on the graphs of both the function and its Taylor polynomials. To get information easily from them, a Taylor approximation $P_n(x_0)$ of $f(x)$ is called "graphically reliable" if both f and P_n graphs coincide at x_0. Besides, an approximation $P_n(x_0)$ of $f(x_0)$ is said to be "more reliable" than $P_m(x_1)$ of $f(x_1)$ if, graphically, $P_n(x_0)$ is nearer to $f(x_0)$ than $P_m(x_1)$ is to $f(x_1)$ (See *Fig. 2*).



$P_3(1)$ and $P_5(1)$ are "graphically reliable"
$P_5(2)$ is reliable, but $P_3(1)$ is not.
$P_5(3)$ is more reliable than $P_5(1)$.

Fig. 2

Though this concept is not rigorous (it depends on the screen resolution, the scale, the observer's sight, ...), it is very useful in order both to introduce, in an intuitive way, the ideas related to the polynomial approach and to work with them.

Some examples are given below.

PROBLEM 1. (Influence of the degree)

Given a function f, students are asked to find an approximation to $f(2)$ by Taylor polynomials at $x = 0$. Varying the degree of the polynomials, they have to observe which of them get a better approximation.

Given $f(x) \quad \sin x$, obtain its Taylor polynomials of degrees 1, 3, 5 and 10 at $x = 0$.

1.1 Is it graphically reliable to approximate $\sin 2$ by the Taylor polynomials of degrees 1 and 3?

1.2 Which of the former polynomials gives a reliable approximation to $\sin 2$?

.3 What is the smallest degree to obtain a reliable approximation to $\sin 2$?

1.4 Give an interval (a, b) such that the third degree Taylor polynomial gets a reliable approximation to $f(x)$ for every $x \in (a, b)$. Do the same for the fifth and the tenth degree Taylor polynomials.

1.5 How does the degree influence the reliability of an approximation?

PROBLEM 2. Influence of the approximated values)

Given a function f and its third degree Taylor polynomial at $x = 0$, students are asked to approximate different values of $f(x)$, to observe the differences about its reliability and to infer some general behavior.

2. Given $g(x) = e^{x/3}$, obtain its second degree Taylor polynomial at $x = 0$.

2.1 Plot both functions.

2.2 Considering the graphs, which approximations are more reliable?

- $e^{2/3}$ or $e^{1/3}$
- $e^{1/3}$ or e
- $e^{1/3}$ or $e^{-1/3}$
- $e^{2/3}$ or $e^{-1/3}$
- $e^{1/3}$ or $e^{-2/3}$

2.3 Order the approximations of $e^{2/3}$, e, $e^{1/3}$, $e^{-2/3}$, e^{-1} and $e^{-1/3}$ according to its reliability.

2.4 How does x's value influence the reliability of an approximation?

PROBLEM 3. (Function's influence)

Given two functions, f and g, students are asked to approximate both $f(2)$ and $g(2)$ by the corresponding third degree Taylor polynomial at $x = 0$ and to observe the differences.

3. Given $f(x) = \sin x$ and $g(x) = e^{x/3}$, obtain its third degree Taylor polynomials at $x = 0$ and plot them.

3.1 Considering the approximations of both $\sin 2$ and $e^{2/3}$, which of them is more reliable?

3.2 Would it be correct to state that the only parameters which the reliability of an approximation depends on are the degree and the x's value? Give another parameter that influences the accuracy of an approximation. (See Lagrange's form of the remainder.)

In every problem, after the study of graphical reliability, we propose to work as well with the numerical values of the approximations. To measure the accuracy of the approximations, students use the classical concept of n-decimals approximation.

As an example, the second part of problem 2 is presented below. Its objective is the students becoming familiar with Lagrange's form of the remainder and its meaning.

2.5 Evaluate the errors made when approaching e and e^{-1} by the second degree Taylor polynomial of $g(x) = e^{x/3}$. Are their signs alike?

2.6 Find the third degree Taylor polynomial, approximate by it both e and e^{-1} and evaluate the errors made. Are their signs alike?

2.7 Considering Lagrage's form of the remainder, what parameters influence the sign of the error?

2.8 Try to give a general rule for the sign of $e^{x/3} - P_n(x)$ depending on the values of n and x.

2.9 Test the rule graphically, graphing different Taylor polynomials of g.

2.10 Find an approximate value for both e and e^{-1} with one-decimal approximation. What degree is needed to obtain e with one-decimal approximation? What about e^{-1}?.

2.11 Does the influence of the x's value on the accuracy of the approximation by Taylor polynomials depend only on the distance from x to 0?

2.3 Approaching roots.

Approaching roots of nonlinear equations is one of the classic topics in Numerical Analysis. Among the approximation methods that are explained in a first course on this subject, there are some with a very clear geometrical meaning, for instance "regula falsi", Newton's method and fixed–point iteration.

To illustrate these methods with both graphical and programming capabilities of Derive, we have designed a practice on this topic.

For each method, the script leads the student to solve one problem, getting approximated values to the roots numerically and graphically.

The general structure of the problems posed is as follows:

Given a function, students first have to locate its roots into intervals of length 1, and to study whether the conditions of convergence for the method are satisfied.

After that, they compute the approximated values to the roots using `ITERATES` and, finally, they are asked to study the graphical meaning of the iterations they have made.

The problem posed to apply the "regula falsi" method is given below:

1.1 Given $f(x) = x^3 + xe^x - 1$, plot it and find intervals of length 1 containing each root of $f(x)$.

1.2 Are the conditions of convergence satisfied? (Study the sign of the second derivative.)

1.3 Define the function:

$$\text{SECLINE}(x, a, b) := f(a) + \frac{x-a}{b-a}(f(b) \quad f(a))$$

which denotes the secant line that joins $(a, f(a))$ and $(b, f(b))$, $a < b$.

1.4 Find and plot the secant line that connects the limits of the interval you have found.

1.5 Find the intersection point of that secant and the x-axis

1.6 Find a general expression, CUT(a, b), for the intersection point of the x-axis and SECLINE(x, a, b).

1.7 Choose the limit of the interval that is convenient to be fixed for doing the iteration. Compute 4 approximated values to the root using `ITERATES`. (Evaluate with `approX`.)

1.8 Plot the secants corresponding to the approximations got in 1.7.

1.9 Use `ITERATES` without specifying the number of iterations and plot the secant corresponding to the last value.

1.10 Give a graphical approximated solution by moving the cross.

It is interesting also to show some cases where the method works though the conditions of convergence are not satisfied, and some cases where the method fails. In the problem about Newton's method we introduce such an example:

2.8 Given $h(x) \quad x \quad \sin x + 0.5$, plot it and find an interval of length 1 containing its root.

2.9 Verify the conditions of convergence for that interval.

2.10 Using `ITERATES`, find the root.

2.11 Apply Newton's method taking two points away from the interval as starting points. Is the method converging in these cases? Were the conditions of convergence satisfied? Is it a contradiction?

2.12 Take $x = 0$ as starting point and iterate three times. What happens? What do you have to take into account when applying Newton's method?

3. EVALUATION OF THE RESULTS

At the end of the academic year 1990-91 we evaluated the improvements of our teaching quality by introducing the use of computing tools.

Both a qualitative and a quantitative analysis were done. The first was based on the results of an inquiry made to the students, and the second on their performance in the course compared with that of the previous academic courses.

From the qualitative analysis we would emphasize the great acceptance that practical lessons, and specially Derive, had among students: 85% of them considered the introduction of computers in teaching very useful, but they appreciated the learning of computing tools more than either the use of a new methodology or the major understanding of the subject. However, they also appreciated that point, marking 4 (on average) out of 5 when asked about the comprehension of graphical representation of functions.

Another issue to be pointed out is that nearly 50% of the inquired had used Derive apart from the practical lessons, and not only to solve questions related to mathematical subjects.

Two factors that, according to the inquiries and our observations, would improve efficiency in those kind of experiences are both the synchronization of prac-

tical lessons with the theoretical ones, and a careful design of the activities and problems posed.

Refering to quantitative evaluation, we have analysed: first, the evolution of the students' performance in the last four years on the subject Math I (include Algebra and Calculus); secondly, the performance of the people who did the practices compared to that of those who did not, but only referred to Calculus.

Considering the percentages of students who passed the subject in the last four years, we observed a very significative increase in the course 90-91 (nearly a 50% of increase).

Considering the marks they got, we had also a quite good improvement.

As a second analysis, we studied the performance of Calculus students in the academic year 1990-91, making a distinction between those who did the practices and those who did not. (From now on we will call them PRACTICE and NO PRACTICE, respectively.) PRACTICE's performance is remarkably high (60% succeded), especially if it is compared to either NO PRACTICE's (28%) or the whole group's (35%).

It might be assumed that people of PRACTICE are the students who are usually interested on the subject: they work (and do the practices) and, as a result, they succeed. That may be right but, analyzing NO PRACTICE's performance (28%) and comparing it to academic year 89-90 results (25%), we might infer that the effect of practices has been to increase the number of students that follow the subject and do try to succeed (one of our big problems is the high percentage of people that do not take any exam at all along the course).

In fact, we could not assure that practices have a direct influence on students' performance; however, they increase a good deal their motivation towards mathematical subjects which influences positively their performance.

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