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The Riemann-Integral and DERIVE, an attempt

I want to show in my lecture some possibilities to visualize and to explore integrating using DERIVE's graphic and calculating abilities. I will try this without using the "antiderivative" at the beginning. I'm sure that all the modules I've prepared can be improved and altered in several ways. You may use them all or only few of them, how you like. You can work with them as a "black box" for the stuents or you can develop them step by step. That will depend on the students' skill and experience in the use of DERIVE.

I call the main- or "motherfile" RIEMANN.MTH. It consists of all the expressions #1 - #21 in the first listing.

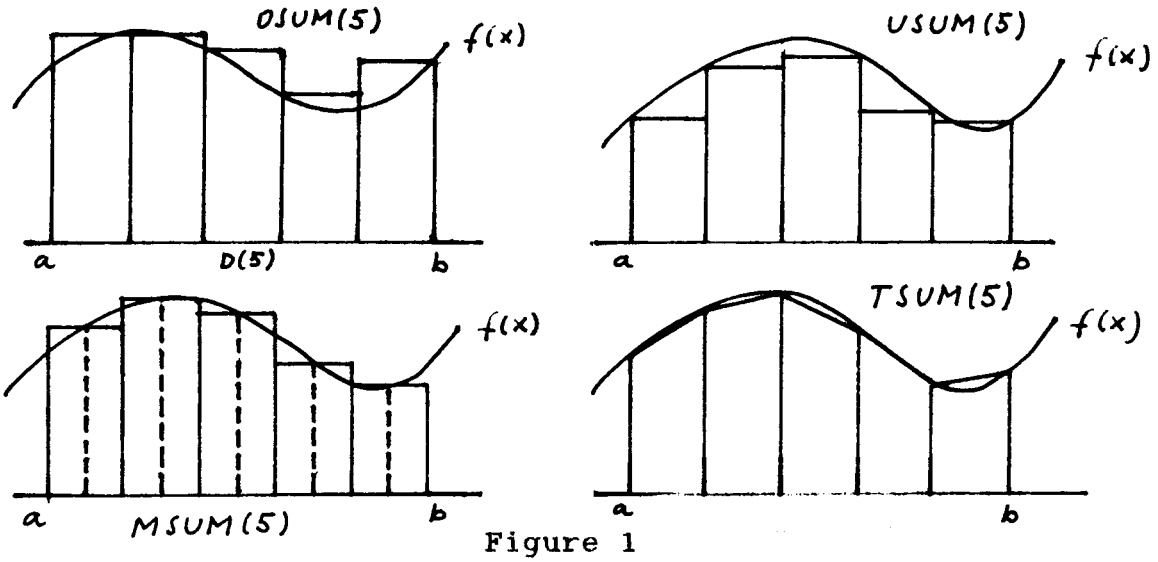


Figure 1

It must be said, that OSUM(n) is not the exact uppersum, and USUM(n) is not the exact lowersum, when the function f(x) is not monotonous. You can observe this, when you take a non-monotonous function and use a little n (n = 2 or 3). I believe, that the file is self-explaining. With COMP_RIEM (expression #21) you can compare the convergence of the four methods of decomposition.

Let work your students with some well known functions.

At the end of this session I tell my students that there is a mathematical function called "defined integral" which calculates under some conditions the area under the functiongraph. And then I let do DERIVE this work (see #45). It's remarkable, that we are able to author the functions and the limits in one vector (see #22).

22: $[F(x) := x^3 - 4x^2 + 4x + 1,\ a := 0,\ b := 3]$

23: OSUM (10) --> approXimate and Plot (Figure 2)

24: OSUM_VAL (10)

25: 6.069

26: USUM (10) --> approXimate and Plot (Figure 3)

27: USUM_VAL (10)

28: 4.476

29: MSUM (10) --> approXimate and Plot (Figure 4)

30: MSUM_VAL (10)

31: 5.23874

32: TSUM (10) --> approXimate and Plot (Figure 5)

33: TSUM_VAL (10)

34: 5.2725

35: MSUM (20) --> approXimate and Plot

36: MSUM (50) --> approXimate and Plot (Figure 6)

37: MSUM (100) --> approXimate and Plot

39: MSUM_VAL (100)

40: 5.24988

43: COMP_RIEM (10, 510, 50)

44:

10	4.476	6.069	5.2725	5.23875
60	5.11639	5.38485	5.25062	5.24968
110	5.17696	5.32340	5.25018	5.24990
160	5.19974	5.30043	5.25008	5.24995
210	5.21169	5.28841	5.25005	5.24997
260	5.21905	5.28101	5.25003	5.24998
310	5.22403	5.27600	5.25002	5.24998
360	5.22764	5.27239	5.25001	5.24999
410	5.23036	5.26966	5.25001	5.24999
460	5.23249	5.26752	5.25001	5.24999
510	5.23421	5.26580	5.25000	5.24999

45: $\int_a^b F(x)\,dx$

46: 5.25

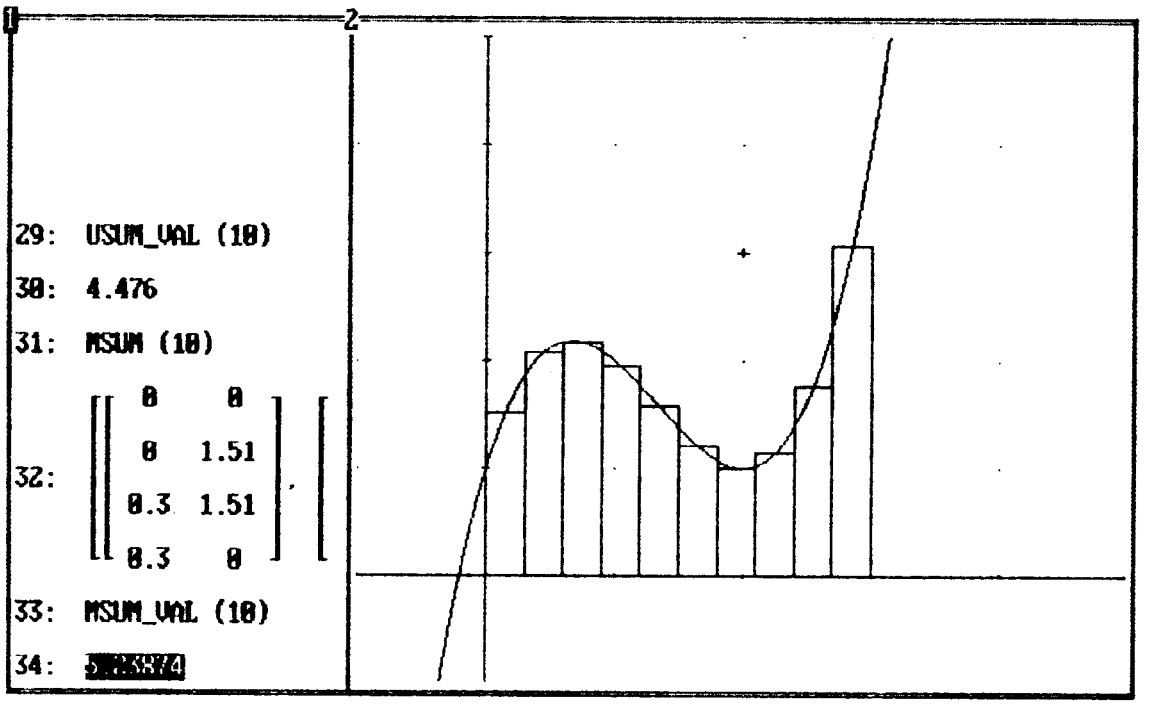


Figure 4

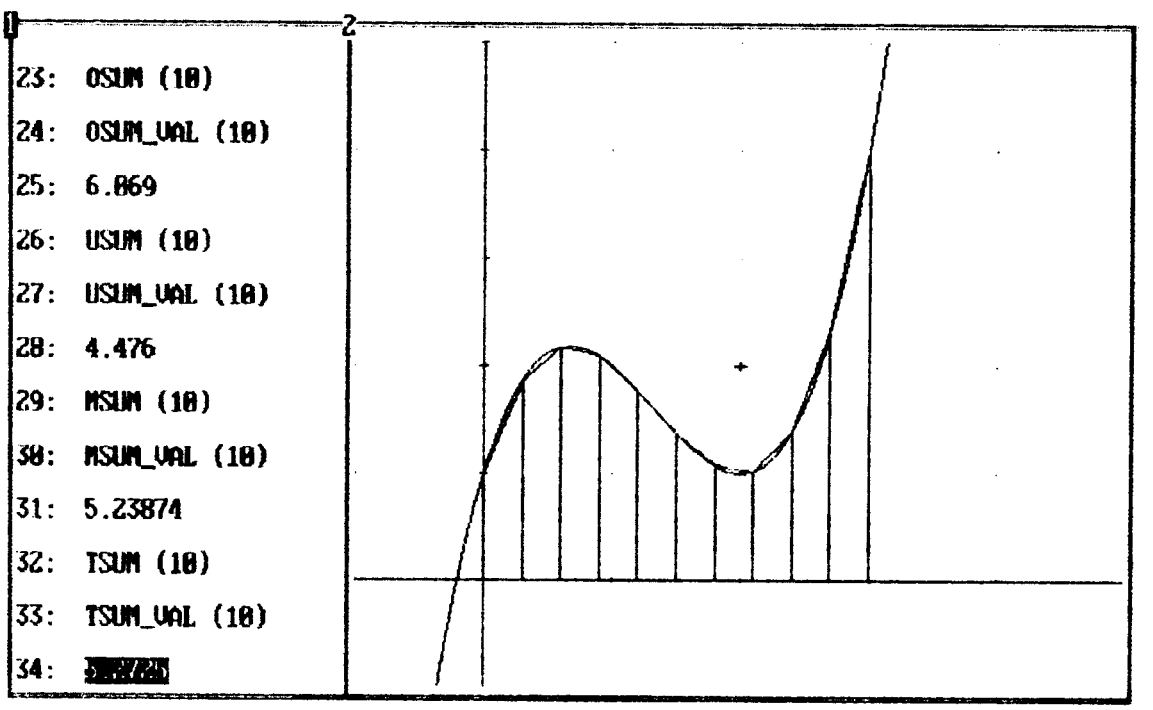


Figure 5

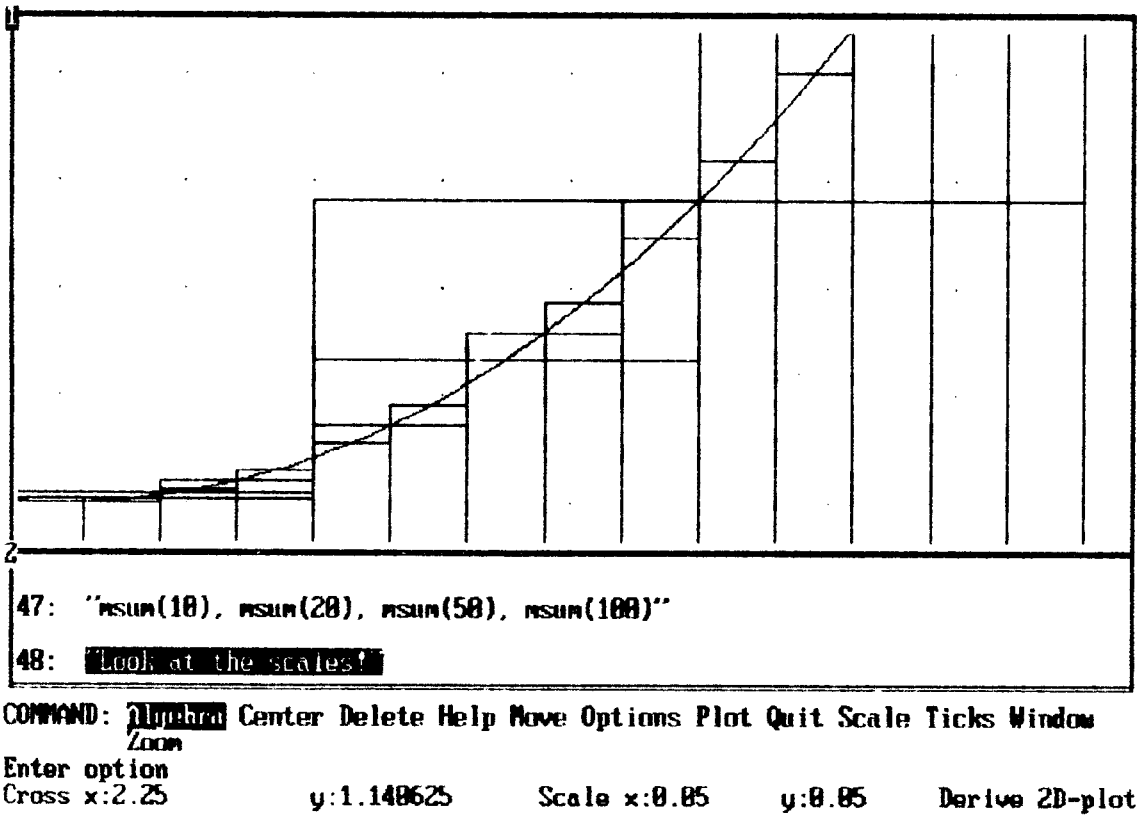


Figure 8

In RIEMO2.MTH we explore what's happening when there are intersections with the x-axis within the limits [a,b] or what's going on, when there is an argument with an undefinded function value. See the different behaviour of the area's values for the functions in #49 and #60! Compare the function plots!

In RIEMO3.MTH I generalize the problem, first by calculating the limes for n to infinity, then by fixing the special limits as a and b. USUM_VAL and OSUM_VAL don't work because of the MAX- and MIN-function. I change to RSUM_VAL(n) and LSUM_VAL(n). Maybe the students can do this work.

RIEMO4.MTH leads to the rules for integrating $c.x^k$, $k \in N$.

First load RIEMANN.MTH

RIEMO2.MTH

22: [F (x) := SIN (2 x), a := 0, b := 2 π]

23: MSUM_VAL (50)

24: 0

25: MSUM (50)

26: "approXimate and Plot !!"

27: F (x) := |SIN (2 x)|

28: MSUM_VAL (50)

29: 3.99473

30: MSUM_VAL (100)

31: 4.00263

32: MSUM_VAL (200)

33: 4.00065

34: $F(x) := \frac{x + 1}{x^2 - 4}$

35: a := 1

36: b := 4

37: MSUM (20)

38: "approXimate and Plot

39: MSUM_VAL (20)

40: -3.38844

41: MSUM_VAL (50)

42: -3.38798

43: MSUM_VAL (100)

44: 4.77417

45: MSUM_VAL (15)

46: 0.692183

47: OSUM_VAL (15)

48: ?

49: $F(x) := \frac{x + 1}{(x - 2)^2}$

50: MSUM_VAL (10)

51: 396.44

52: MSUM_VAL (50)

53: 1964.66

54: MSUM_VAL (15)

55: 144.247

56: USUM_VAL (15)

57: 45.8463

58: OSUM_VAL (15)

59: ∞

60: $F(x) := \frac{x + 1}{2 (x - 2)^{2/3}}$

61: OSUM_VAL (20)

62: 10.1286

63: OSUM_VAL (15)

64: ∞

65: MSUM_VAL (20)

66: 9.49213

67: MSUM_VAL (100)

68: 10.0171

69: MSUM_VAL (200)

70: 10.1646

71: MSUM_VAL (500)

72: 10.3162

73: MSUM_VAL (1000)

74: 10.4036

75: $\int_1^4 F(x)\, dx$

76: $\frac{21\ 2^{1/3}}{4} + \frac{33}{8}$

77: 10.7395

First load RIEMANN.MTH

RIEMO3.MTH

22: $\left[F(x) := \frac{x^2+1}{2},\ a := -1,\ b := 2.5\right]$

23: MSUM_VAL (n)

24: $\frac{7\,(124\,n^2 - 49)}{192\,n^2}$

25: $\lim_{n \to \infty} \frac{7\,(124\,n^2 - 49)}{192\,n^2}$

26: $\frac{217}{48}$

27: TSUM_VAL (n)

28: $\frac{7\,(62\,n^2 + 49)}{96\,n^2}$

29: $\lim_{n \to \infty} \frac{7\,(62\,n^2 + 49)}{96\,n^2}$

30: $\frac{217}{48}$

31: USUM_VAL (n)

32: $\frac{7\,(62\,n^2 + 49)}{96\,n^2} - \frac{49 \sum_{k=1}^{n} |14\,k - 4\,n - 7|}{32\,n^3}$

33: $\lim_{n \to \infty} \left[\frac{7\,(62\,n^2 + 49)}{96\,n^2} - \frac{49 \sum_{k=1}^{n} |14\,k - 4\,n - 7|}{32\,n^3}\right]$

34: "cannot be calculated! Why not?"

35: "We change USUM_VAL and OSUM_VAL to LSUM_VAL and RSUM_VAL"

36: LSUM (n) := VECTOR $\left[\begin{bmatrix} XL(n) & 0 \\ XL(n) & F(XL(n)) \\ XR(n) & F(XL(n)) \\ XR(n) & 0 \end{bmatrix},\ k,\ 1,\ n\right]$

37: LSUM (10)

38: "approXimate and Plot!"

39: LSUM_VAL (n) := $\sum_{k=1}^{n} D(n)\,F(XL(n))$

40: LSUM_VAL (n)

41: $\frac{7\,(62\,n^2 - 63\,n + 49)}{96\,n^2}$

42: $\lim_{n \to \infty} \frac{7\,(62\,n^2 - 63\,n + 49)}{96\,n^2}$

43: $-\frac{217}{48}$

44: $\text{RSUM}(n) := \text{VECTOR}\left[\begin{bmatrix} \text{XL}(n) & 0 \\ \text{XL}(n) & \text{F}(\text{XR}(n)) \\ \text{XR}(n) & \text{F}(\text{XR}(n)) \\ \text{XR}(n) & 0 \end{bmatrix}, k, 1, n\right]$

45: $\text{RSUM_VAL}(n) := \sum_{k=1}^{n} \text{D}(n)\,\text{F}(\text{XR}(n))$

46: $\lim_{n \to \infty} \text{RSUM_VAL}(n)$

47: $-\frac{217}{48}$

62: "now with general limits a and b"

49: a := a

50: b := b

51: $\lim_{n \to \infty}$ [MSUM_VAL (n), TSUM_VAL (n), RSUM_VAL (n), LSUM_VAL (n)]

52: $\left[\frac{(b-a)\,(a^2 + a\,b + b^2 + 3)}{6}, \frac{(b-a)\,(a^2 + a\,b + b^2 + 3)}{6}, \frac{(b-a)\,(a^2 + a\,b + b^2 + 3)}{6}, \frac{(b-a)\,(a^2 + a\,b + b^2 + 3)}{6}\right]$

53: $\left[-\frac{a^3}{6} - \frac{a}{2} + \frac{b^3}{6} + \frac{b}{2}, -\frac{a^3}{6} - \frac{a}{2} + \frac{b^3}{6} + \frac{b}{2}, -\frac{a^3}{6} - \frac{a}{2} + \frac{b^3}{6} + \frac{b}{2}, -\frac{a^3}{6} - \frac{a}{2} + \frac{b^3}{6} + \frac{b}{2}\right]$

63: "Manage Substitute for a=-1 ; b=2.5; Ctrl ENTER!"

55: $\left[-\frac{217}{48}, \frac{217}{48}, \frac{217}{48}, \frac{217}{48}\right]$

64: "there exists a mathematical function, which we want to find!"

56: $\int_a^b \text{F}(x)\,dx$

57: $-\frac{a^3 + 3\,a - b\,(b^2 + 3)}{6}$

58: $-\frac{a^3}{6} - \frac{a}{2} + \frac{b^3}{6} + \frac{b}{2}$

LSUM(n), RSUM(n), LSUM_VAL(n) and RSUM_VAL(n) should be taken up into RIEMANN.MTH for further use!

Some rules to calculate the area under the functiongraph for important functions ($f(x) = c.x^k$).

First load RIEMANN.MTH

RIEMO4.MTH

46: "Elementary rules for the defined integral"

47: "f(x) = c x^k for k = 1,2,3,4,5,6,...."

48: F (x) := c x

49: $\lim_{n \to \infty}$ MSUM_VAL (n)

50: $\frac{c\,(b - a)\,(a + b)}{2}$

51: $\frac{b^2\,c}{2} - \frac{a^2\,c}{2}$

52: F (x) := $c\,x^2$

53: $\lim_{n \to \infty}$ MSUM_VAL (n)

54: $\frac{c\,(b - a)\,(a^2 + a\,b + b^2)}{3}$

55: $\frac{b^3\,c}{3} - \frac{a^3\,c}{3}$

56: F (x) := $c\,x^3$

57: $\lim_{n \to \infty}$ TSUM_VAL (n)

58: $\frac{c\,(b - a)\,(a^3 + a^2\,b + a\,b^2 + b^3)}{4}$

59: $\frac{b^4\,c}{4} - \frac{a^4\,c}{4}$

60: F (x) := $c\,x^7$

61: $\lim_{n \to \infty}$ MSUM_VAL (n)

62: "323 sec"

63: $\frac{c\,(b - a)(a^7 + a^6\,b + a^5\,b^2 + a^4\,b^3 + a^3\,b^4 + a^2\,b^5 + a\,b^6 + b^7)}{8}$

64: $\frac{b^8\,c}{8} - \frac{a^8\,c}{8}$

65: F (x) := $c\,x^{-3}$

66: MSUM_VAL (n)

67: $$8\,c\,n^2\,(a-b)\sum_{k=1}^{n}\frac{1}{(2\,k\,(a-b)-a\,(2\,n+1)+b)^3}$$

68: $$\lim_{n\to\infty} 8\,c\,n^2\,(a-b)\sum_{k=1}^{n}\frac{1}{(2\,k\,(a-b)-a\,(2\,n+1)+b)^3}$$

69: $F(x) := \sqrt{x}$

70: MSUM_VAL (n)

71: $$\frac{\sqrt{2}\,(b-a)\sum_{k=1}^{n}\sqrt{\left[-\frac{2\,k\,(a-b)-a\,(2\,n+1)+b}{n}\right]}}{2\,n}$$

72: $$\lim_{n\to\infty}\frac{\sqrt{2}\,(b-a)\sum_{k=1}^{n}\sqrt{\left[-\frac{2\,k\,(a-b)-a\,(2\,n+1)+b}{n}\right]}}{2\,n}$$

For $f(x) := x^k$ integrating in this way only works for $k \in N$

In RIEMO5.MTH we try another decomposition of the interval [a,b]: (b - a) is the sum of an infinite geometric series:

$$b - a = \frac{a_1}{1-q} \qquad 0 < q < 1$$

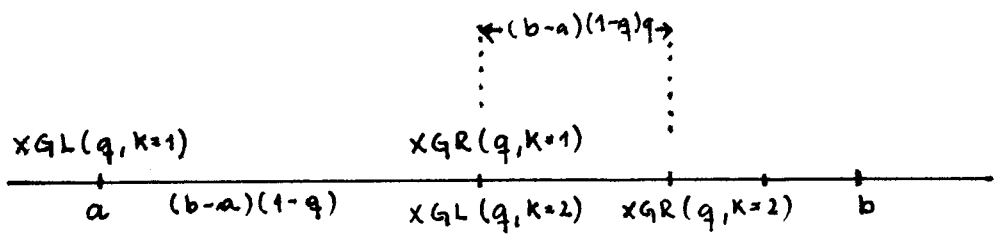


Figure 9

RIEMO5.MTH

1: $$\text{GSUM_VAL}\ (q,\ n) := (b-a)\left[\frac{1}{q}-1\right]\sum_{k=1}^{n} q^k\,F\,(b+(a-b)\,q^k)$$

2: $\text{XGL}\ (q) := b - (b-a)\,q^{k-1}$

3: $\text{XGR}\ (q) := b - (b-a)\,q^{k}$

4: $$\text{GSUM}\ (n,\ q) := \text{VECTOR}\left[\begin{bmatrix} \text{XGL}\ (q) & 0 \\ \text{XGL}\ (q) & F\ (\text{XGL}\ (q)) \\ \text{XGR}\ (q) & F\ (\text{XGR}\ (q)) \\ \text{XGR}\ (q) & 0 \end{bmatrix},\ k,\ 1,\ n\right]$$

5: $[F\ (x) := x^2,\ a := 1,\ b := 4]$

6: $\text{GSUM_VAL}\ \left[\frac{1}{2},\ \infty\right]$

7: 27.8571

8: GSUM_VAL (0.75, ∞)

9: 23.7104

10: GSUM_VAL (0.99, ∞)

11: 21.0909

12: GSUM_VAL (q, ∞)

13: ?

14: "We have to declare 0 < q < 1 !! Then Simplify once more"

15: $$\frac{3\,(q^3 + 17\,q^2 + 8\,q + 16)}{(q^2 + q + 1)\,(q + 1)}$$

16: $$\lim_{q \to 1} \frac{3\,(q^3 + 17\,q^2 + 8\,q + 16)}{(q^2 + q + 1)\,(q + 1)}$$

17: 21

18: "general limits a and b"

19: [a := a, b := b]

20: GSUM_VAL (q, ∞)

21: $$\frac{(b - a)\,(a^2\,q^2\,(q + 1) + 2\,a\,b\,q + b^2\,(q^2 + 1))}{(q^2 + q + 1)\,(q + 1)}$$

22: $$\lim_{q \to 1} \frac{(b - a)\,(a^2\,q^2\,(q + 1) + 2\,a\,b\,q + b^2\,(q^2 + 1))}{(q^2 + q + 1)\,(q + 1)}$$

23: $$\frac{(b - a)\,(a^2 + a\,b + b^2)}{3}$$

24: $$\frac{b^3}{3} - \frac{a^3}{3}$$

25: $$\left[F\,(x) := \frac{x^2 + 1}{2},\ a := -1,\ b := 3\right]$$

26: GSUM (10, 0.5)

27: "approXimate and Plot! --> Figure 10"

28: GSUM_VAL (0.5, 10)

29: 8.55191

30: GSUM (50, 0.9)

31: "approXimate and Plot! --> Figure 11"

32: GSUM_VAL (0.9, 50)

33: 6.72526

34: GSUM_VAL (0.99, 1000)

35: 6.67938

36: GSUM_VAL (0.99, ∞)

37: 6.68025

38: GSUM_VAL (q, ∞)

39: $$\frac{4\,(q^3 + 6\,q^2 - 2\,q + 5)}{(q^2 + q + 1)\,(q + 1)}$$

40: $$\lim_{q \to 1} \frac{4\,(q^3 + 6\,q^2 - 2\,q + 5)}{(q^2 + q + 1)\,(q + 1)}$$

41: $$\frac{20}{3}$$

42: $$\int_a^b F\,(x)\,dx$$

43: $$\frac{20}{3}$$

44: "Let's try for f(x) = c x^5"

45: $[F\,(x) := c\,x^5,\ a := a,\ b := b]$

46: GSUM_VAL (q, ∞)

47: "Simplify this and then calculate the limit for q --> inf"

48: $$\frac{c\,(b - a)\,(a^5 + a^4\,b + a^3\,b^2 + a^2\,b^3 + a\,b^4 + b^5)}{6}$$

49: $$\frac{b^6\,c}{6} - \frac{a^6\,c}{6}$$

50: "Can you see the rule?"

51: "Now let's try for f(x) = √x and for f(x) = 1/x !"

52: F (x) := √x

53: GSUM_VAL (q, ∞)

54: $$\frac{(q - 1)\,(a - b)\,\sum_{k=1}^{\infty} q^k\,\sqrt{(q^k\,(a - b) + b)}}{q}$$

55: $$\lim_{q \to 1} \frac{(q - 1)\,(a - b)\,\sum_{k=1}^{\infty} q^k\,\sqrt{(q^k\,(a - b) + b)}}{q}$$

56: "this expression cannot be calculated"

57: $F\,(x) := \frac{1}{x}$

58: GSUM_VAL (q, ∞)

59: $$\frac{(q - 1)\,(a - b)\,\sum_{k=1}^{\infty} \frac{q^k}{q^k\,(a - b) + b}}{q}$$

60: "Derive is not able to evaluate this sum, too. Are you able to?"

61: "So we have to try another kind of decomposition."

GSUM(10,0.5)

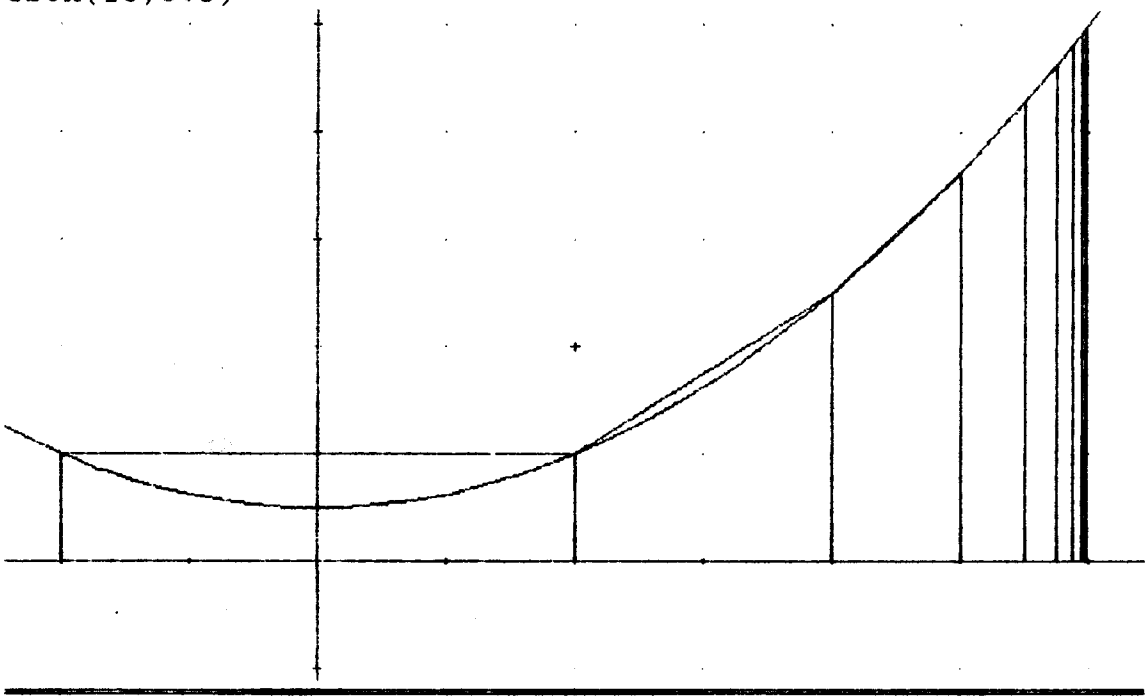
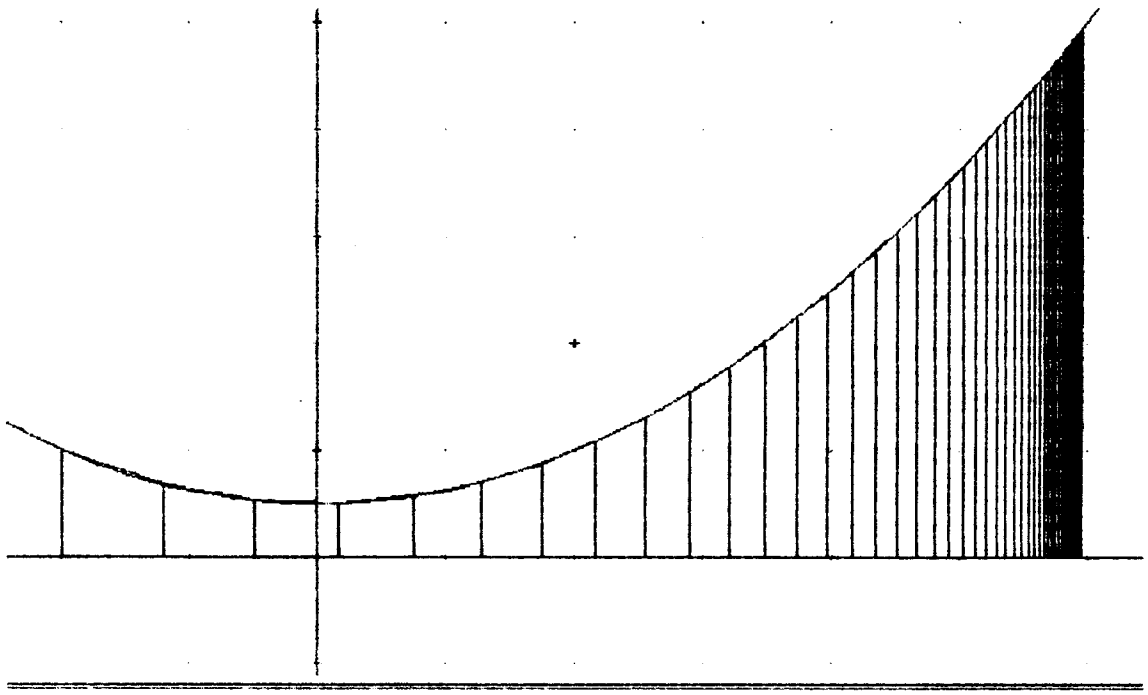


Figure 10

GSUM(50,0.9)



COMMAND: [redacted] Center Delete Help Move Options Plot Quit Scale Ticks Window
Zoom
Enter option
Cross x:1 y:2 Scale x:0.5 y:1 Derive 2D-plot

Figure 11

In RIEM06.MTH I try to find a function for the area under the functiongraph for some important and wellknown functions. The upper limit is now a variable x. So we obtain the area as a function of the upper limit. At this point I show the first time the connection to the "antiderivative" of a function. The students build the 1st derivative of the areafunction and .....

First load RIEMANN.MTH

RIEM06.MTH

30: F (x) := SIN (x)

31: a := a

32: b := b

33: MSUM_VAL (n)

34: $$-\frac{0.5\ (a - b)\ \mathrm{COS}\ (a)}{n\ \mathrm{SIN}\left[\frac{0.5\ a}{n} - \frac{0.5\ b}{n}\right]} + \frac{0.5\ (b - a)\ \mathrm{COS}\ (b)}{n\ \mathrm{SIN}\left[\frac{0.5\ a}{n} - \frac{0.5\ b}{n}\right]}$$

35: "Use Factor!!"

36: $$-\frac{0.5\ (a - b)\ (\mathrm{COS}\ (a) - \mathrm{COS}\ (b))}{n\ \mathrm{SIN}\left[\frac{0.5\ a}{n} - \frac{0.5\ b}{n}\right]}$$

37: $$\lim_{n \to \infty} \frac{0.5\ (a - b)\ (\mathrm{COS}\ (a) - \mathrm{COS}\ (b))}{n\ \mathrm{SIN}\left[\frac{0.5\ a}{n} - \frac{0.5\ b}{n}\right]}$$

38: COS (a) - COS (b)

39: "The Area under the exponentialfunction!"

40: F (x) := $\hat{e}^{x}$

41: MSUM (n)

42: MSUM_VAL (n)

43: $$\frac{\hat{e}^{a(0.5/n+1) + 0.5b/n}(b - a)}{n\ (\hat{e}^{a/n} - \hat{e}^{b/n})} + \frac{\hat{e}^{0.5a/n + b(0.5/n+1)}(a - b)}{n\ (\hat{e}^{a/n} - \hat{e}^{b/n})}$$

44: $$\frac{\hat{e}^{0.5\ a\ /\ n + 0.5\ b\ /\ n}\ (b - a)\ (\hat{e}^{a} - \hat{e}^{b})}{n\ (\hat{e}^{a/n} - \hat{e}^{b/n})}$$

45: $$\lim_{n \to \infty} \frac{\hat{e}^{0.5\ a\ /\ n + 0.5\ b\ /\ n}\ (b - a)\ (\hat{e}^{a} - \hat{e}^{b})}{- \hat{e}^{b/n})}$$

46: $\hat{e}^{b} - \hat{e}^{a}$

47: $F(x) := \frac{x^3}{3} - x^2 + \frac{1}{2}$

48: $a := a$

49: $b := x$

50: $AR(x) := \lim_{n \to \infty} MSUM_VAL(n)$

51: $\frac{(x - a)(x^3 + x^2(a - 4) + a\,x(a - 4) + a^3 - 4a^2 + 6)}{12}$

52: $\frac{x^4}{12} - \frac{x^3}{3} + \frac{x}{2} - \frac{a^4}{12} + \frac{a^3}{3} - \frac{a}{2}$

53: $FL(x) := \frac{x^4}{12} - \frac{x^3}{3} + \frac{x}{2} - \frac{a^4}{12} + \frac{a^3}{3} - \frac{a}{2}$

54: $FL(4) - FL(1)$

55: $\frac{7}{4}$

56: $a := 1$

57: $b := 4$

58: $\lim_{n \to \infty} MSUM_VAL(n)$

59: $\frac{7}{4}$

60: $\frac{d}{dx} FL(x)$

61: $\frac{x^3}{3} - x^2 + \frac{1}{2}$

62: $a := c$

63: $b := x$

64: $F(x) := COS\left[-\frac{x}{4}\right]$

65: $MSUM_VAL(n)$

66: $\frac{(x - c)\,SIN\left[\frac{x}{4}\right]}{2\,n\,SIN\left[\frac{x}{8\,n} - \frac{c}{8\,n}\right]} + \frac{(c - x)\,SIN\left[\frac{c}{4}\right]}{2\,n\,SIN\left[\frac{x}{8\,n} - \frac{c}{8\,n}\right]}$

67: $\frac{(x - c)\left[SIN\left[\frac{x}{4}\right] - SIN\left[\frac{c}{4}\right]\right]}{2\,n\,SIN\left[\frac{x}{8\,n} - \frac{c}{8\,n}\right]}$

68: $\lim_{n \to \infty} \frac{(x - c)\left[SIN\left[\frac{x}{4}\right] - SIN\left[\frac{c}{4}\right]\right]}{2\,n\,SIN\left[\frac{x}{8\,n} - \frac{c}{8\,n}\right]}$

69: $4\,SIN\left[\frac{x}{4}\right] - 4\,SIN\left[\frac{c}{4}\right]$

70: $\frac{d}{dx}\left[4 \text{ SIN}\left[\frac{x}{4}\right] - 4 \text{ SIN}\left[\frac{c}{4}\right]\right]$

71: $\text{COS}\left[\frac{x}{4}\right]$

RIEM07.MTH makes it possible to work with $f(x) = x^k$ without the restriction $k \in N$. [a,b] is decomposited so that the divisionpoints build a geometric sequence. $a \neq 0$

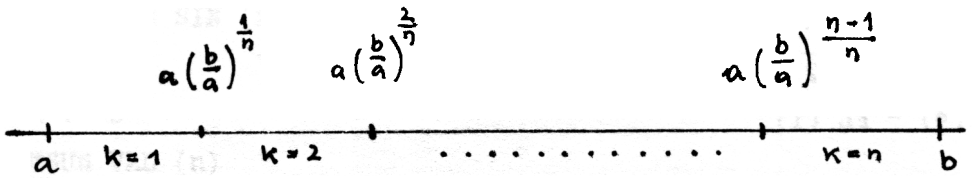


Figure 12

RIEM07.MTH

1: "The decomposition is a finite geometric sequence!

2: $\text{XFGL (n)} := a\left[\frac{b}{a}\right]^{(k-1)/n}$

3: $\text{XFGR (n)} := \text{XFGL (n)}\left[\frac{b}{a}\right]^{1/n}$

4: $\text{FGLSUM (n)} := \text{VECTOR}\left[\begin{bmatrix} \text{XFGL (n)} & 0 \\ \text{XFGL (n)} & \text{F (XFGL (n))} \\ \text{XFGR (n)} & \text{F (XFGL (n))} \\ \text{XFGR (n)} & 0 \end{bmatrix}, k, 1, n\right]$

5: $\left[F(x) := \frac{x^2}{2} + \frac{1}{2},\ a := 1,\ b := 4\right]$

6: FGLSUM (10)

7: "approXimate and Plot"

8: FGLSUM (50)

9: "approXimate and Plot - see figure 13!"

10: $\text{FGLSUM_VAL (n)} := \sum_{k=1}^{n} (\text{XFGR (n)} - \text{XFGL (n)})\ \text{F (XFGL (n))}$

11: FGLSUM_VAL (10)

12: 10.5825

13: FGLSUM_VAL (50)

14: 11.7102

15: FGLSUM_VAL (100)

16: 11.8547

17: FGLSUM_VAL (n)

18: $\frac{3\ (2^{4/n} + 2^{2/n} + 22)}{2\ (2^{4/n} + 2^{2/n} + 1)}$

19: $\lim_{n \to \infty} \frac{3\ (2^{4/n} + 2^{2/n} + 22)}{2\ (2^{4/n} + 2^{2/n} + 1)}$

20: 12

21: [a := a, b := b]

22: FGLSUM_VAL (n)

23: $\frac{(b - a)\ \left[\left[\frac{b}{a}\right]^{2/n} + \left[\frac{b}{a}\right]^{1/n} + a^2 + a\ b + b^2 + 1\right]}{2\ \left[\left[\frac{b}{a}\right]^{2/n} + \left[\frac{b}{a}\right]^{1/n} + 1\right]}$

24: $\lim_{n \to \infty} \frac{(b - a)\ \left[\left[\frac{b}{a}\right]^{2/n} + \left[\frac{b}{a}\right]^{1/n} + a^2 + a\ b + b^2 + 1\right]}{2\ \left[\left[\frac{b}{a}\right]^{2/n} + \left[\frac{b}{a}\right]^{1/n} + 1\right]}$

25: $\frac{(b - a)\ (a^2 + a\ b + b^2 + 3)}{6}$

26: $-\frac{a^3}{6} - \frac{a}{2} + \frac{b^3}{6} + \frac{b}{2}$

27: "Now let's try once more with a negative power!"

28: $F\ (x) := x^{-2}$

29: FGLSUM_VAL (n)

30: $\left[\frac{1}{a} - \frac{1}{b}\right]\ \left[\frac{b}{a}\right]^{1/n}$

31: $\lim_{n \to \infty} \left[\frac{1}{a} - \frac{1}{b}\right]\ \left[\frac{b}{a}\right]^{1/n}$

32: $\frac{1}{a} - \frac{1}{b}$

33: "Oh' it's working now!"

34: $F\ (x) := \frac{1}{x}$

35: FGLSUM_VAL (n)

36: $n\ \left[\frac{b}{a}\right]^{1/n} - n$

37: $\lim_{n \to \infty} \left[n\ \left[\frac{b}{a}\right]^{1/n} - n\right]$

38: $LN\ \left[\frac{b}{a}\right]$

39: [F (x) := $\sqrt{x}$, a := a, b := b]

40: FGLSUM_VAL (n)

41: $a \sum_{k=1}^{n} \sqrt{\left[a \left[\frac{b}{a}\right]^{k/n - 1/n}\right]} \left[\frac{b}{a}\right]^{k/n} - a \left[\sum_{k=1}^{n} \sqrt{\left[a \left[\frac{b}{a}\right]^{k/n - 1/n}\right]} \left[\frac{b}{a}\right]^{k/n}\right] \left[\frac{b}{a}\right]^{-1/n}$

42: $\lim_{n \to \infty} \left[a \sum_{k=1}^{n} \sqrt{\left[a \left[\frac{b}{a}\right]^{k/n - 1/n}\right]} \left[\frac{b}{a}\right]^{k/n} - a \left[\sum_{k=1}^{n} \sqrt{\left[a \left[\frac{b}{a}\right]^{k/n - 1/n}\right]} \left[\frac{b}{a}\right]^{k/n}\right] \left[\frac{b}{a}\right]^{-1/n}\right]$

43: "Declare all variables nonnegative!!"

44: $\frac{2 (b^{3/2} - a^{3/2})}{3}$

45: "the function of the area for x^(-1), √x, x^(-3/4), x^m, ..."

46: $\left[F(x) := \frac{1}{x},\ a := c,\ b := x\right]$

47: FGLSUM_VAL (n)

48: $n \left[\frac{x}{c}\right]^{1/n} - n$

49: $\lim_{n \to \infty} \left[n \left[\frac{x}{c}\right]^{1/n} - n\right]$

50: $\mathrm{LN} \left[\frac{x}{c}\right]$

51: $F(x) := \sqrt{x}$

52: $c \sum_{k=1}^{n} \sqrt{\left[c \left[\frac{x}{c}\right]^{k/n - 1/n}\right]} \left[\frac{x}{c}\right]^{k/n} - c \left[\sum_{k=1}^{n} \sqrt{\left[c \left[\frac{x}{c}\right]^{k/n - 1/n}\right]} \left[\frac{x}{c}\right]^{k/n}\right] \left[\frac{x}{c}\right]^{-1/n}$

53: "Declare Variables nonnegative!"

54: $\frac{c^{1/(2n)} (x^{3/2} - c^{3/2}) (x^{1/(2n)} + c^{1/(2n)})}{x^{1/n} + (c\,x)^{1/(2n)} + c^{1/n}}$

55: $\lim_{n \to \infty} \frac{c^{1/(2n)} (x^{3/2} - c^{3/2}) (x^{1/(2n)} + c^{1/(2n)})}{x^{1/n} + (c\,x)^{1/(2n)} + c^{1/n}}$

56: $\frac{2 (x^{3/2} - c^{3/2})}{3}$

57: $F(x) := x^{-3/4}$

58: $(x^{1/4} - c^{1/4}) \left[\frac{x}{c}\right]^{3/(4n)} + (x^{1/4} - c^{1/4}) \left[\frac{x}{c}\right]^{1/(2n)} + (x^{1/4} - c^{1/4}) \left[\frac{x}{c}\right]^{1/(4n)} + x^{1/4} - c^{1/4}$

59: $\lim_{n \to \infty} \left[(x^{1/4} - c^{1/4}) \left[\frac{x}{c}\right]^{3/(4n)} + (x^{1/4} - c^{1/4}) \left[\frac{x}{c}\right]^{1/(2n)} + (x^{1/4} - c^{1/4}) \left[\frac{x}{c}\right]^{1/(4n)} + x^{1/4} - c^{1/4}\right]$

60: $4\,x^{1/4} - 4\,c^{1/4}$

61: $F\,(x) := m\,x^{p/q}$

62: $$\frac{m\,c^{p\,/\,(n\,q)}\,(x^{p/q\,+\,1} - c^{p/q\,+\,1})\,(x^{1/n} - c^{1/n})}{x^{(p/q\,+\,1)\,/\,n} - c^{(p/q\,+\,1)\,/\,n}}$$

63: $$\lim_{n \to \infty} \frac{m\,c^{p\,/\,(n\,q)}\,(x^{p/q\,+\,1} - c^{p/q\,+\,1})\,(x^{1/n} - c^{1/n})}{x^{(p/q\,+\,1)\,/\,n} - c^{(p/q\,+\,1)\,/\,n}}$$

64: $$\frac{m\,q\,x^{p/q\,+\,1}}{p + q} - \frac{m\,q\,c^{p/q\,+\,1}}{p + q}$$

65: $$\frac{m\,q\,(x^{p/q\,+\,1} - c^{p/q\,+\,1})}{p + q}$$

And that's finally the integration rule for the power function without any restrictions.

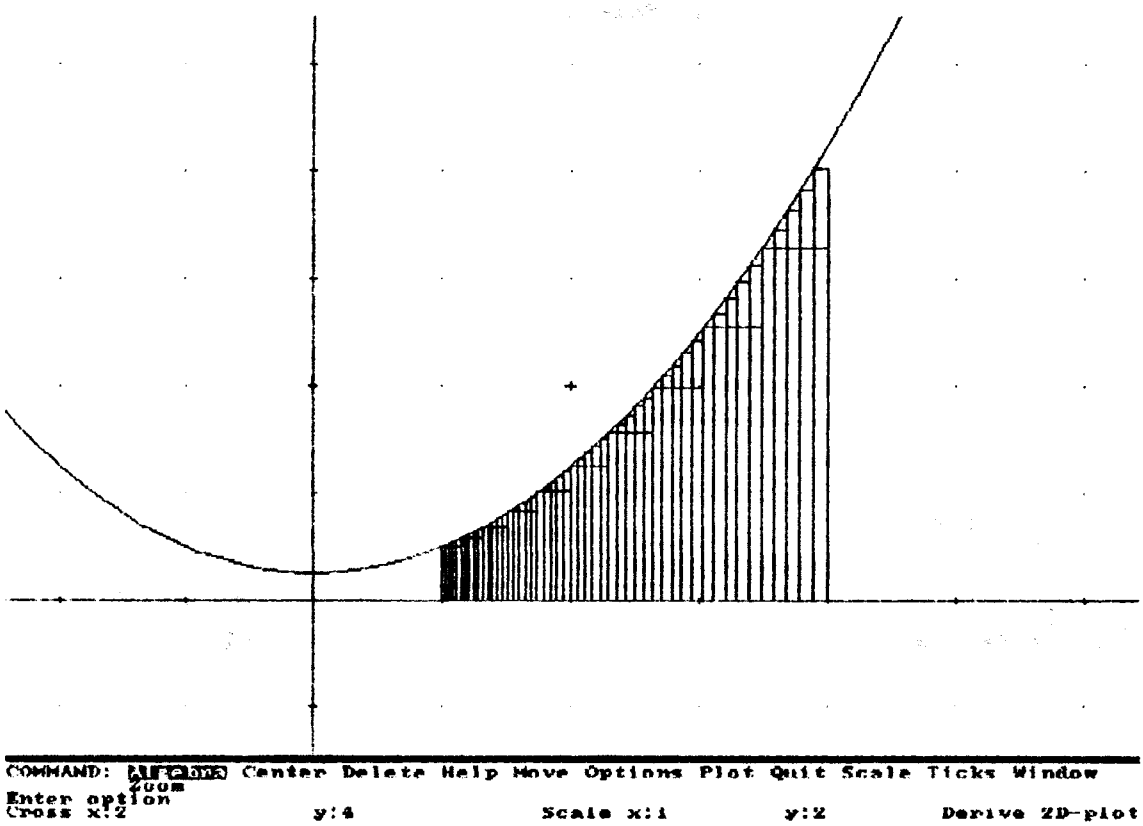


Figure 13

In RIEM08.MTH I want to demonstrate how to calculate the area between the graphs of two functions f(x) and g(x).

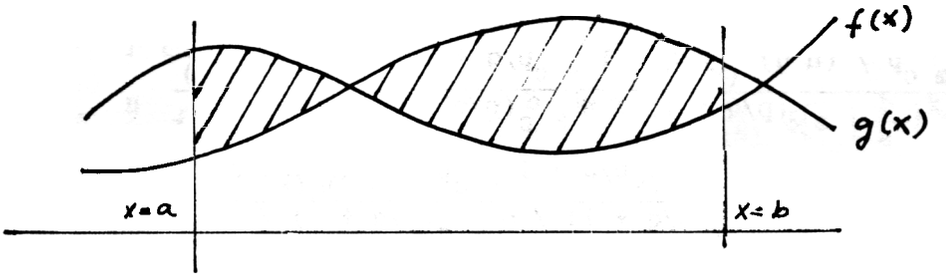


Figure 14

RIEM08.MTH

1: a :=

2: b :=

3: F (x) :=

4: G (x) :=

5: $D(n) := \frac{b - a}{n}$

6: XL (n) := a + (k - 1) D (n)

7: XR (n) := a + k D (n)

8: XM (n) := a + (k - 0.5) D (n)

9: FMID (n) := F (XM (n))

10: $MSUM_VAL(n) := \sum_{k=1}^{n} D(n)\ FMID(n)$

11 $TSUM2(n) := VECTOR\left[\begin{bmatrix} XL(n) & F(XL(n)) \\ XL(n) & G(XL(n)) \\ XR(n) & G(XR(n)) \\ XR(n) & F(XR(n)) \\ XL(n) & F(XL(n)) \end{bmatrix}, k, 1, n\right]$

12: $TSUM2_VAL(n) := \sum_{k=1}^{n} \frac{D(n)\ (G(XR(n)) - F(XR(n)) + n\sim}{2}$

$\sim)) - F(XL(n))))$

13: $\left[F(x) := \frac{x^2}{4},\ G(x) := -\frac{x^2}{4} + \frac{x}{2} + \frac{3}{2},\ a := -1,\right.$

14: TSUM2 (20)

15: "Simplify and Plot the approximating trapeziums!"

16: TSUM2_VAL (20)

17: 3.74437

18: TSUM2 (50)

19: "Simplify and Plot in another colour!"

20: TSUM2_VAL (50)

21: 3.7491

22: TSUM2_VAL (n)

23: $\frac{3\,(5\,n^2 - 3)}{4\,n^2}$

24: $\lim_{n \to \infty} \frac{3\,(5\,n^2 - 3)}{4\,n^2}$

25: 3.75

26: [a := -3, b := 3]

27: TSUM2 (50)

28: "approXimate and Plot! - see figure 16!"

29: TSUM2_VAL (50)

30: -0.0072

31: "What's the reason for this strange result?"

32: F (x) = G (x)

33: $x = 0.5 - 0.5\sqrt{13}$

34: $x = 0.5\sqrt{13} + 0.5$

35: "We will work with a function FLAECHE for the area:"

36: [a := a, b := b]

37: $\lim_{n \to \infty}$ TSUM2_VAL (n)

38: $\frac{2\,a^3 - 3\,a^2 - 18\,a - 2\,b^3 + 3\,b^2 + 18\,b}{12}$

39: FLAECHE (a, b) := $\frac{2\,a^3 - 3\,a^2 - 18\,a - 2\,b^3 + 3\,b^2}{12}$ +

40: FLAECHE $(-3,\ 0.5 - 0.5\sqrt{13})$

41: $-\frac{13\sqrt{13}}{24} - \frac{35}{24}$

42: FLAECHE $(0.5 - 0.5\sqrt{13},\ 0.5\sqrt{13} + 0.5)$

43: $\frac{13\sqrt{13}}{12}$

44: FLAECHE $(0.5\sqrt{13} + 0.5,\ 3)$

45: $\frac{35}{24} - \frac{13\sqrt{13}}{24}$

46: $\left|-\frac{13\sqrt{13}}{24} - \frac{35}{24}\right| + \left|\frac{13\sqrt{13}}{12}\right| + \left|\frac{35}{24} - \frac{13\sqrt{13}}{24}\right|$

47: $\frac{13\sqrt{13}}{6}$

48: 7.81202

49: "Now we'll try another way to find the area:"

50: F (x) - G (x)

51: $\dfrac{x^2 - x - 3}{2}$

52: $F(x) := \dfrac{x^2 - x - 3}{2}$

53: $\lim_{n \to \infty} \text{MSUM_VAL}(n)$

54: $\dfrac{(b - a)(2a^2 + a(2b - 3) + 2b^2 - 3b - 18)}{12}$

55: "Compare with the expression in #38! - Conclusions?"

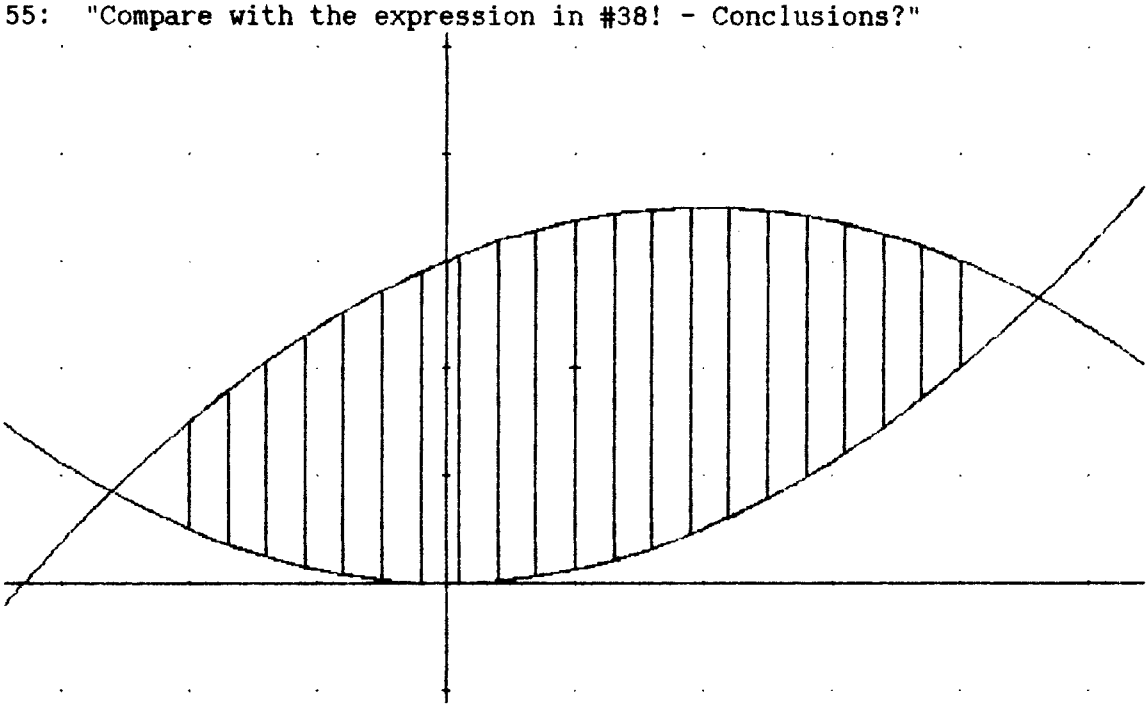


Figure 15

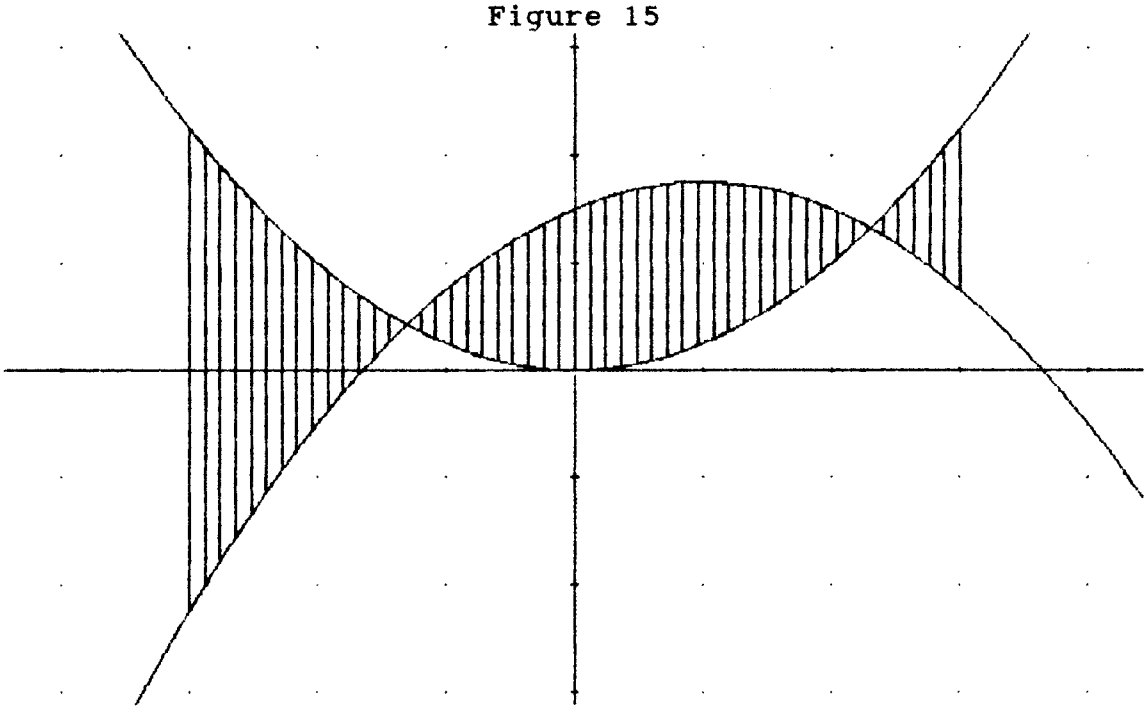


Figure 16

RIEM09.MTH is possible with DERIVE version 2.10 or higher, because I need random numbers. I think the file and my sketch are explaining the procedure: the 1st decomposition dec have to be authored, the rest is done automatically step by step:

pt_sequ <Simplify>

p_s := #(of the pt_sequ)

dsum <approXimate and Plot>

dsum_val <approXimate>

x_half <Simplify>

d_new <Simplify>

dec := #(of d_new); then pt_sequ <Simplify>

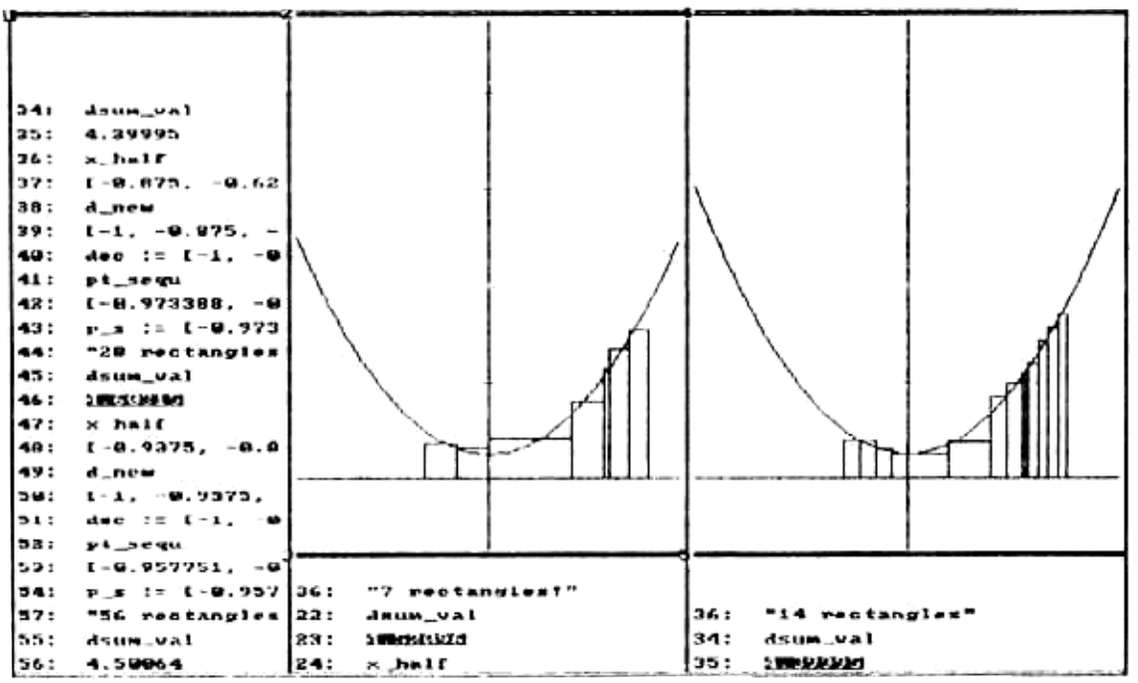


Figure 17

RIEMO8.MTH

1: $\left[F(x) := \frac{x^2}{2} + \frac{1}{2},\ a := -1,\ b := 2.5\right]$

2: dec := []

3: p_s := []

4: n := DIMENSION (dec) - 1

5: XE (k) := ELEMENT (dec, k)

6: DIST (k) := XE (k + 1) - XE (k)

7: pt_sequ := VECTOR (XE (k) + DIST (k) RANDOM (1), k, 1, n)

8: XI (k) := ELEMENT (p_s, k)

9: dsum_val := $\sum_{k=1}^{n}$ DIST (k) F (XI (k))

dsum := VECTOR $\left[\begin{bmatrix} XE(k) & 0 \\ XE(k) & F(XI(k)) \\ XE(k+1) & F(XI(k)) \\ XE(k+1) & 0 \end{bmatrix}, k, 1, n\right]$

11: x_half := VECTOR $\left[\frac{XE(k-1) + XE(k)}{2}\ k, 2, n+1\right]$

12: d_new := APPEND $\left[[ELEMENT(dec, 1)], VECTOR\left[IF\left[FLOOR\left[\frac{k}{2}\right] /= \frac{k}{2}, ELEMENT\left[x_half, \frac{k+1}{2}\right], ELEMENT\left[dec, \frac{k+2}{2}\right]\right], k, 1, 2n\right]\right]$

13: "We define a decomposition:"

14: dec := [-1, -0.5, 0, 1.3, 1.8, 1.9, 2.2, 2.5]

15: "pt_sequ produces random points in the intervalls"

16: pt_sequ

17: [-0.617451, -0.112747, 0.454593, 1.69255, 1.86108, 1.94854, 2.30096]

18: "p_s is the sequence of points: p_s := #16"

19: p_s := [-0.617451, -0.112747, 0.454593, 1.69255, 1.86108, 1.94854, 2.30096]

20: dsum

21: "Simplify and Plot"

22: dsum_val

23: 4.23587

24: "I bisect the intervals"

25: x_half

26: [-0.75, -0.25, 0.65, 1.55, 1.85, 2.05, 2.35]

```
27:  "That's the new decomposition:"
28:  d_new
29:  [-1, -0.75, -0.5, -0.25, 0, 0.65, 1.3, 1.55, 1.8, 1.85, 1.9, 2.~
     ~05, 2.2, 2.35, 2.5]
30:  dec := [-1, -0.75, -0.5, -0.25, 0, 0.65, 1.3, 1.55, 1.8,~
     ~ 1.85, 1.9, 2.05, 2.2, 2.35, 2.5]
31:  pt_sequ
32:  [-0.867628, -0.586834, -0.291636, -0.0108941, 0.200560, 1.12856~
     ~, 1.38412, 1.55302, 1.82139, 1.89023, 1.94104, 2.10492, 2.24945~
     ~, 2.42710]
33:  p_s := [-0.867628, -0.586834, -0.291636, -0.0108941, 0.200560, ~
     ~1.12856, 1.38412, 1.55302, 1.82139, 1.89023, 1.94104, 2.10492, ~
     ~2.24945, 2.42710]
34:  dsum
35:  "Simplify and Plot!"
36:  dsum_val
37:  4.47422
38:  x_half
39:  [-0.875, -0.625, -0.375, -0.125, 0.325, 0.975, 1.425, 1.675, 1.~
     ~825, 1.875, 1.975, 2.125, 2.275, 2.425]
40:  d_new
41:  [-1, -0.875, -0.75, -0.625, -0.5, -0.375, -0.25, -0.125, 0, 0.3~
     ~25, 0.65, 0.975, 1.3, 1.425, 1.55, 1.675, 1.8, 1.825, 1.85, 1.8~
     ~75, 1.9, 1.975, 2.05, 2.125, 2.2, 2.275, 2.35, 2.425, 2.5]
42:  dec := [-1, -0.875, -0.75, -0.625, .........
     ~425, 2.5]
43:  pt_sequ
44:  [-0.954644, -0.818638, -0.682074, -0.538200, ............
     ~43, 2.39116, 2.47952]
45:  p_s := [-0.954644, -0.818638, -0.682074, -0.538200, .........
     ~, 2.34043, 2.39116, 2.47952]
46:  dsum
47:  "Simplify and Plot!"
48:  dsum_val
49:  4.52085
50:  x_half
51:  [ ............. ]
```

```
52:  d_new
53:  "56 parts"
54:  [ -1, ................, 2.5]
55:  dec := [ -1, ................, 2.5]
56:  pt_sequ
57:  [-0.942841, -0.883283, -0.822853, -0.783464, -0.689425, .... ]
58:  p_s := [-0.942841, -0.883283, -0.822853, -0.783464, .... ]
59:  dsum
60:  dsum_val
61:  4.49643
62:  "the last approximation was better!"
```

63: $\int_a^b F(x)\,dx$

```
64:  4.52083
65:  x_half
66:  [-0.96875, ............, 2.48125]
67:  d_new
68:  [-1, -0.96875, ....... 2.48125, 2.5]
69:  dec := [-1, -0.96875, ....... 2.48125, 2.5]
70:  pt_sequ
71:  [-0.995810, -0.967200, -0.916693, ........, 2.49294]
72:  p_s := [-0.995810, -0.967200, -0.916693, ........, 2.49294]
73:  DIMENSION (p_s)
74:  112
75:  dsum_val
76:  4.51777
77:  "the approximations are oscillating round the exact value!"
78:  "Now try with any other decomposition of the starting interval!
```

The figures are from another run, that's the reason for the different numerical values.

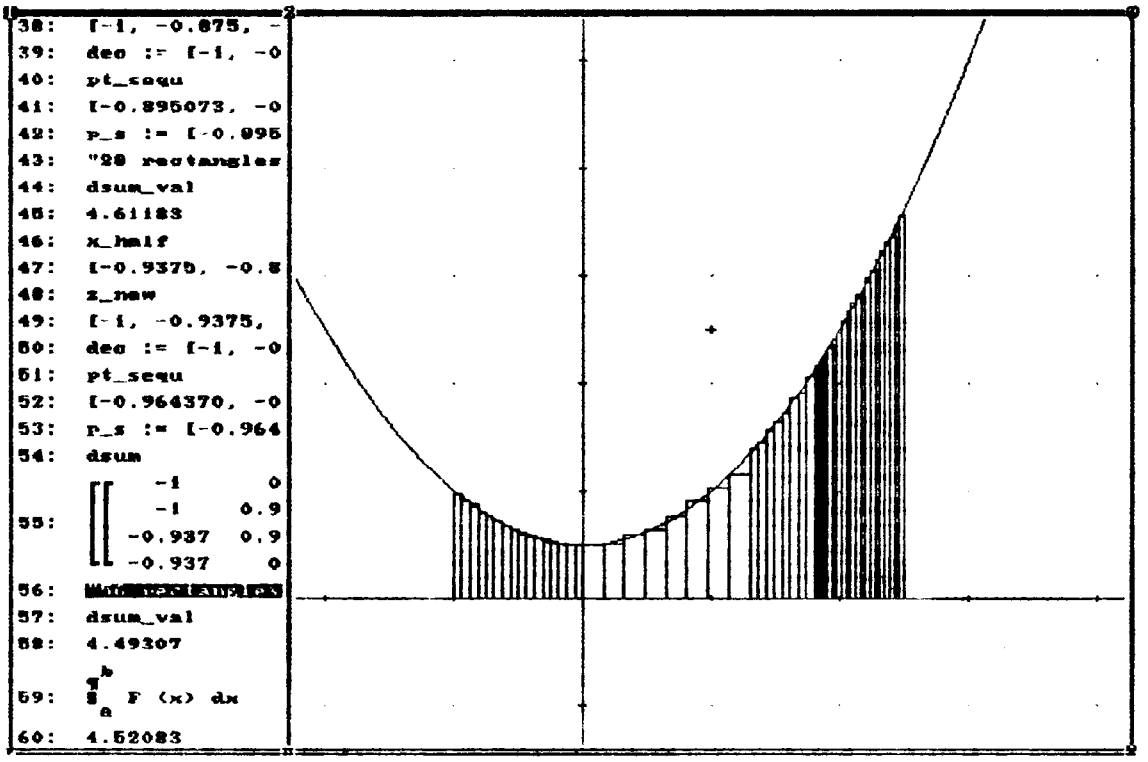


Figure 18

RIEM10.MTH demonstrates the calculating of volumes generated by revolving curves about the x-axis and RIEM11.MTH shows how to find the area between a curve and the y-axis.

First load RIEMANN.MTH

RIEM10.MTH

22: $\text{VSUM}(n) := \text{VECTOR}\left[\begin{bmatrix} XL(n) & -FMID(n) \\ XL(n) & FMID(n) \\ XR(n) & FMID(n) \\ XR(n) & -FMID(n) \\ XL(n) & -FMID(n) \end{bmatrix}, k, 1, n\right]$

23: $\text{VSUM_VAL}(n) := \sum_{k=1}^{n} FMID(n)^2 \, \pi \, D(n)$

24: CONTOUR (a, b, x) := [F (x), IF (x ≥ a AND x ≤ b, - F (x), 0), ~
~F (x)]

25: $[F(x) := x^3 - 4x^2 + 4x + 1,\ a := 0,\ b := 3]$

26: CONTOUR (a, b, x)

27: VSUM (10)

28: "Simplify and Plot #26 and #27!"

29: VSUM (50)

30: "Simplify and Plot"

31 VSUM_VAL (10)

32 31.9025

33: VSUM_VAL (50)

34: 32.4257

35: $\pi \int_a^b F(x)^2 \, dx$

36: 32.4481

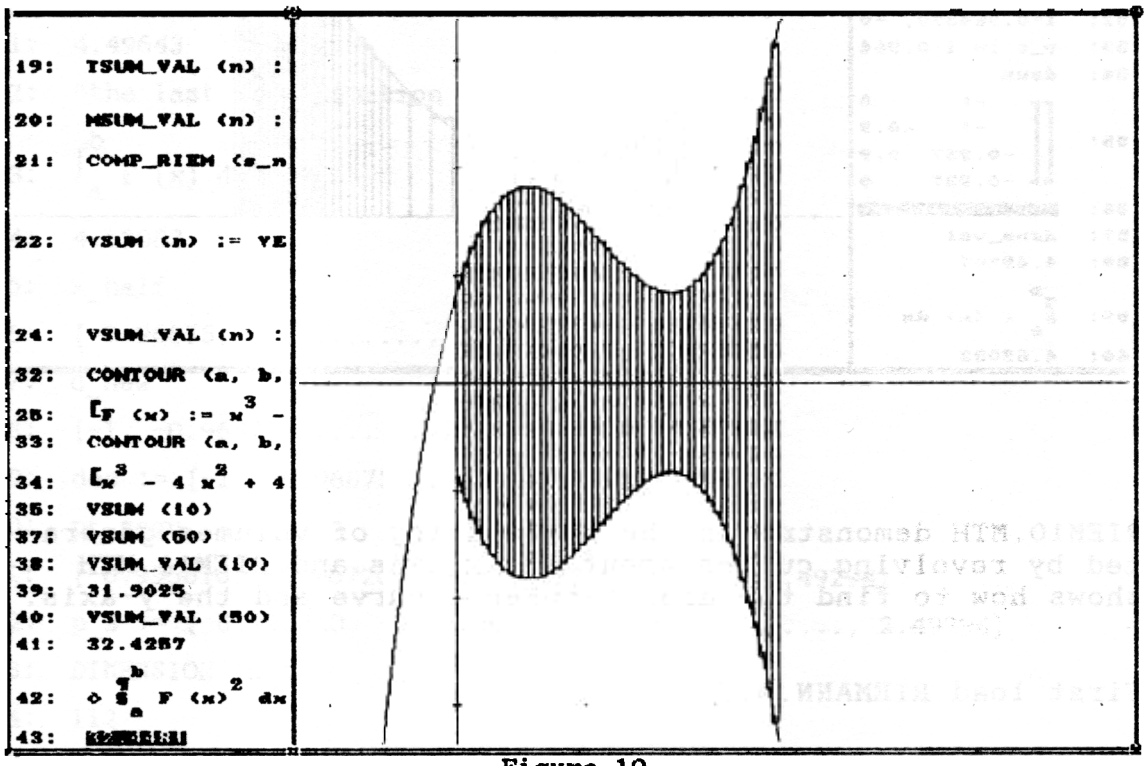


Figure 19

RIEM11.MTH

1: a :=

2: b :=

3: F (x) :=

4: G (y) :=

5: D (n) := $\frac{b - a}{n}$

6: YU (n) := a + (k - 1) D (n)

7: YO (n) := a + k D (n)

8: YM (n) := a + (k - 0.5) D (n)

9: GMID (n) := G (YM (n))

10: MSUM_VAL (n) := $\sum_{k=1}^{n}$ D (n) GMID (n)

11: $\text{MSUM}(n) := \text{VECTOR}\left[\begin{bmatrix} 0 & YU(n) \\ GMID(n) & YU(n) \\ GMID(n) & YO(n) \\ 0 & YO(n) \end{bmatrix}, k, 1, n\right]$

12: $\left[F(x) := \frac{x^2}{2},\ a := 1,\ b := 3\right]$

13: $y = \frac{x^2}{2}$

14: $x = -\sqrt{2}\sqrt{y}$

15: $x = \sqrt{2}\sqrt{y}$

16: $G(y) := \sqrt{2}\sqrt{y}$

17: MSUM (10)

18: "approXimate and Plot!"

19: MSUM_VAL (10)

20: 3.95666

21: VECTOR (MSUM_VAL (k), k, 10, 100, 10)

22: [3.95666, 3.95629, 3.95622, 3.95620, 3.95619, 3.95618, 3.95617,~
~ 3.95617, 3.95617, 3.95617]

23: MSUM_VAL (n)

24: $\frac{2.82842 \sum_{k=1}^{n} \sqrt{(2k + n - 1)}}{n^{1.5}}$

25: $\lim_{n \to \infty} \frac{2.82842 \sum_{k=1}^{n} \sqrt{\left[\frac{2k + n - 1}{n}\right]}}{n}$

26: "the formulae for the area from c to y is:"

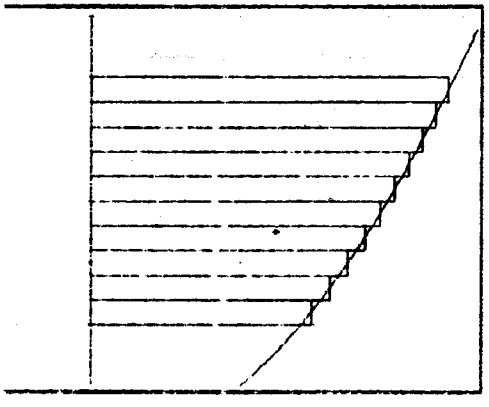
27: $\text{FLAECHE}(y) := \frac{2\sqrt{2}}{3} y^{3/2} + c$

28: "Using this formulae calculate the area from 1 to 3!"

29: FLAECHE (3) - FLAECHE (1)

30: 3.95617

Figure 20



RIEM12.MTH at last will explain the Simpson-approximation:

First load RIEMANN.MTH

RIEM12.MTH

30: SIMP_VAL (n) := $\sum_{k=1}^{n} \frac{D(n)\,(F(XL(n)) + 4\,F(XM(n)) + F(XR(n)))}{6}$

31: SIM_PAR (n) := VECTOR $\left| \begin{bmatrix} x & p\,x^2 + q\,x + r \\ XL(n) & F(XL(n)) \\ XM(n) & F(XM(n)) \\ XR(n) & F(XR(n)) \end{bmatrix}, k\ 1\ n \right|$

32: SIM_CURV (n) := VECTOR (CHI (XL (n), x, XR (n)) ELEMENT (SIM PA`
`R (n), k), k, 1, n)

SIM_STR (n) := VECTOR $\begin{bmatrix} XL(n) & F(XL(n)) \\ XL(n) & 0 \\ XR(n) & 0 \\ XR(n) & F(XR(n)) \end{bmatrix}, k, 1\ n$

34 $\left| F(x) := \frac{x^2 + e^x}{1 + x^2 + SIN(2\,x)} \quad a := -1 \quad b := 3 \right|$

35: SIM_PAR (4)

36: "Simplify and Plot - figure 21"

37: SIM_CURV (4)

38: SIM_STR (4)

39: "Simplify anf Plot #37 and #38 to obtain figure 22"

40: SIMP_VAL (4)

41: 7.64142

42: SIM_PAR (10)

43: SIM_CURV (10)

44: SIM_STR (10)

45: SIMP_VAL (10)

46: 7.50244

"the result, DERIVE gives us:"

48: $\int_a^b F(x)\,dx$

49: 7.50471

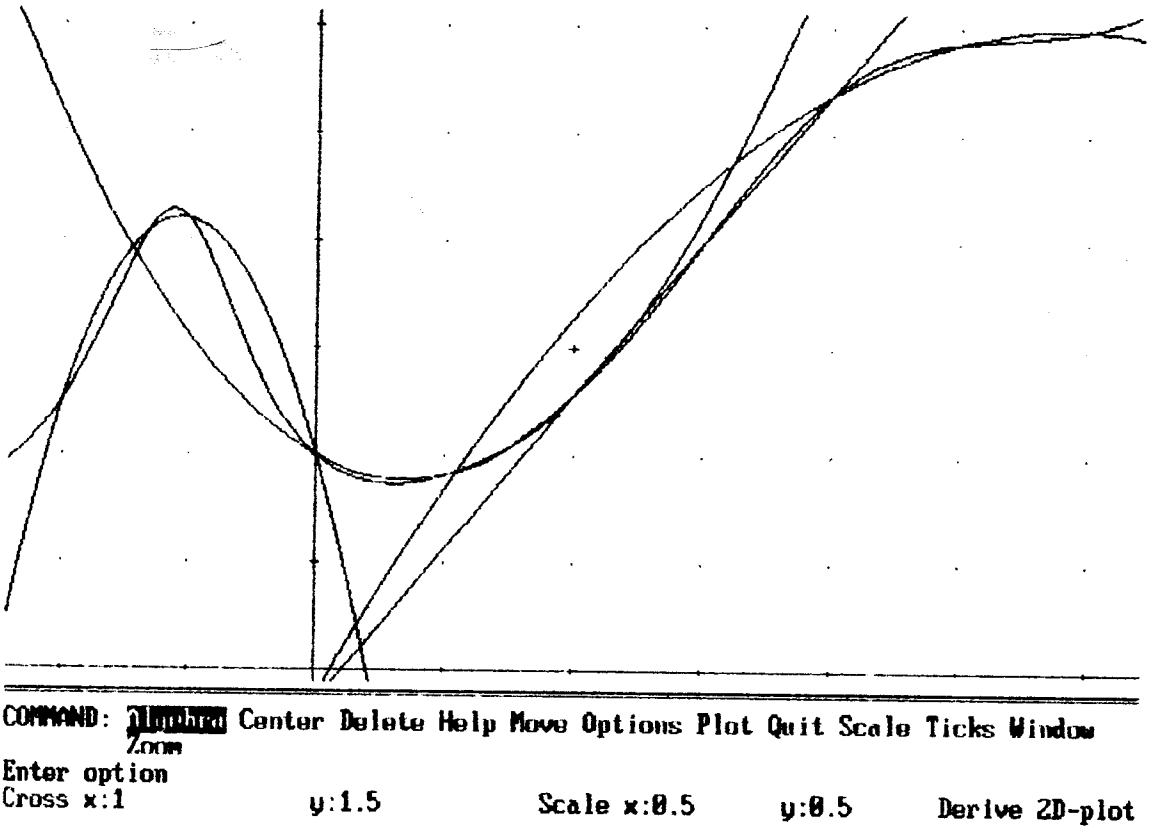


Figure 21

SIMP_VAL(n) returns the numerical value depending on the number n of parts in which [a,b] is divided;

SIM_PAR(n) returns a vector consisting of the approximating parabolas;

SIM_CURV(n) returns another vector consisting of the parabolas' pieces you need to approximate the curve;

SIM_STR(n) produces the vertical lines.

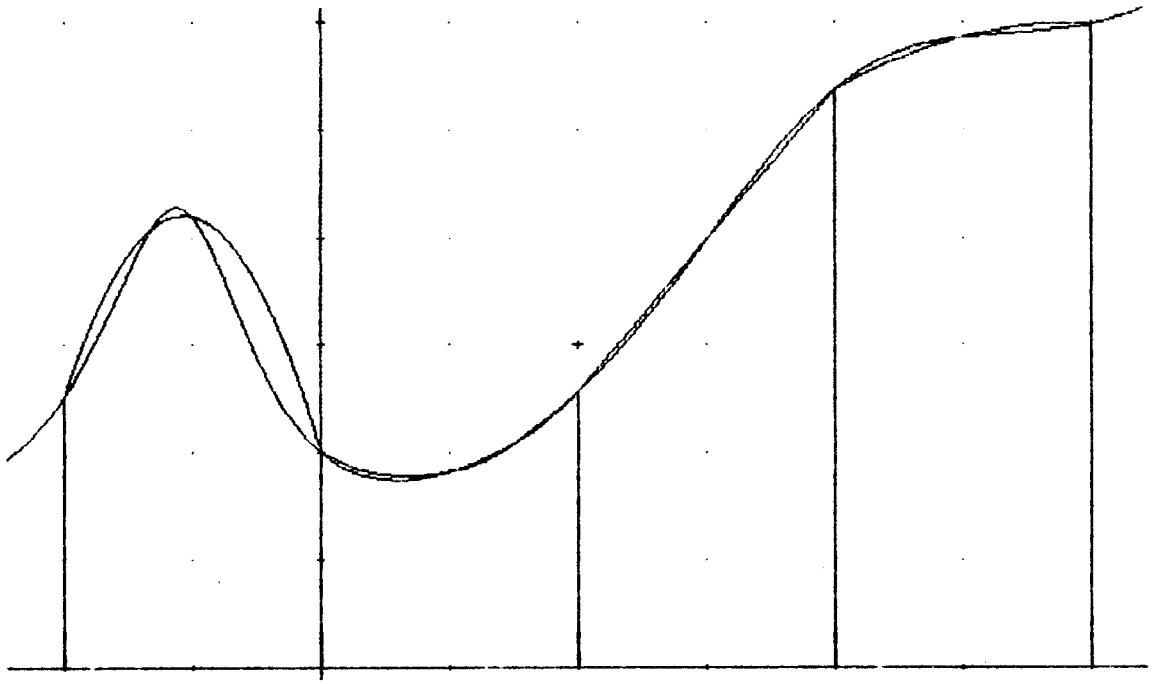


Figure 22

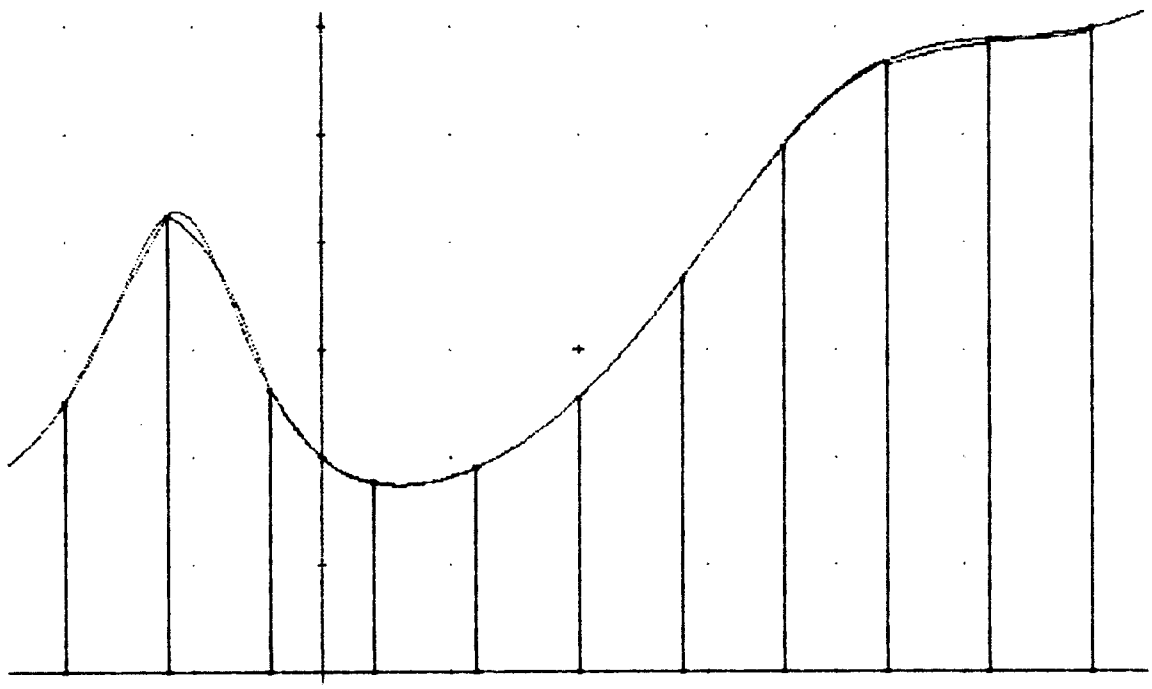


Figure 23