BACK

Introducing Calculus with *DERIVE*

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Abstract: This paper presents an approach to the teaching of calculus using DERIVE at the University of Plymouth in England. The approach is used with students who intend to become engineers but who have neither any previous knowledge of nor any qualifications in advanced mathematics. Many of them are mature students who have returned to study after a considerable gap in their education. Calculus is introduced through computer experiments as are the exponential and natural logarithm functions. Using DERIVE it is possible to lead students fairly quickly to the solution by separation of variables of first order differential equations

INTRODUCTION

The Extended Engineering Degree Course at The University of Plymouth (formerly Plymouth Polytechnic South West) began in September 1990. It grew out of the former Higher Introductory Technology and Engineering Conversion Course (HITECC) which had been offered at the Polytechnic since 1987. The first year provides students with the academic foundations they lack for direct entry onto the degree or diploma courses in engineering taught at Plymouth. Teaching modules consist of mathematics, engineering science, engineering experience and project work.

The mathematics module of the new course comprises 6 hours per week of mathematics teaching and tutorial work for two terms and 6 hours per week of mathematical modelling for one term out of which a project is chosen.

There are two target groups: (1) mature students who have either no Advanced (A) Level backgrounds or limited A level backgrounds in mathematics and physics; (2) school students with weak A-Levels in these subjects. There is no end of year examination. All work is continually assessed through coursework, projects and end of unit tests.

There are currently 85 students working on the course. Subject to the level at which they pass this foundation year, the students proceed to degree or diploma courses in the engineering subject of their choice.

DERIVE and EXTENDED ENGINEERING

Use of the computer algebra package DERIVE has been a conspicuous feature of the course since 1989. A laboratory with 25 workstations (each one an Opus 286) is available for two hours a week when required, bookable in advance. There are four mathematics teaching groups so that it is possible for each student to work individually at a computer. However, it is often desirable for the students to work in pairs.

At Plymouth teaching mathematics with DERIVE usually takes one of three forms: (1) student investigations prior to a formal lecture to motivate a topic from the outset; (2) student investigations following a formal lecture to amplify points made in the lecture; (3) use of a laptop computer and overhead-projector-palette during a formal lecture. When students are working at the computer themselves they usually follow a handout prepared by the lecturer to guide them through the investigation. They are also encouraged to use DERIVE by themselves outside of contact time with staff, both to familiarise themselves with the package and to check work that they do by hand.

During the first month of the course those mathematical skills required by the General Certificate of Secondary Education (GCSE) are reviewed and all done by hand. This is the content of the first mathematics unit. The students sit a test at the end of the unit but the test is zero-weighted in the overall context of the course.

The first time that the students meet DERIVE is, therefore, after they have become familiar with the concepts and processes of elementary mathematics.

Almost the first item in the second unit is the introduction to DERIVE. Students are taught how to drive the package using examples from the material in unit one: **Author** and **Enter** to display expressions on the screen; **Manage Substitute** for the evaluation of algebraic expressions; **Factor** to factorize polynomial expressions; **Expand** in order to expand brackets; **soLve** for the roots of polynomial equations.

DERIVE's easy to handle menu system usually enables the students to master the fundamental processes within an hour. In fact DERIVE is so easy to use that many of our students who were not previously computer-literate often progressively become so through their frequent use of DERIVE. Figure (i) is a sample of the material used in our students first session with the package.

INTRODUCING CALCULUS

This takes the form of an investigation and assumes no prior knowledge other than GCSE mathematics and the ability to handle the package. (For those students who do already have some knowledge of calculus the investigation is offered as revision and for further practice with DERIVE.) By using the commands **Calculus Differentiate Variable:x Order:1** the students are encouraged to look for emerging patterns as powers of x are differentiated by DERIVE. (Figure (ii)).

It is not difficult for them to guess the pattern for positive powers of x. Indeed, some students may well have decided to see what DERIVE would do with x^n. Problems emerge with negative and fractional powers since DERIVE's screen display does not prefer a negative index. (Figures (iii) and (iv)).

Here the students are required to recall the index laws of elementary algebra in order to interpret what they see on the screen. This in itself is a valuable activity and forces the students to question what they are seeing and not to passively accept it.

During an investigation like this the teacher becomes a mobile learning-resource for the students. After an introductory talk of around five minutes to get the class going, the teacher can concentrate on responding to the questions that will naturally arise and the discoveries that are made. It is astonishing quite how much mathematics can be discussed between the students themselves and with the lecturer during one of these sessions.

An interesting development usually takes place during this investigation. Once they have grown confident in their control of the process and have begun to see that

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

the students usually begin to remark, *"OK! I can see what is happening but what is it for?"* At this point the students themselves are expressing a desire to progress from a manipulative skill to the understanding of a concept. They then come to the following (traditional!) lecture about differential calculus actually eager to begin finding out what calculus is for.

We adopt a similar proceedure for introducing integral calculus (figures (v), (vi) and (vii)).

N.B. It would be useful if DERIVE could provide the arbitrary constant of indefinite integration. When writing students are prone to leave it out no matter how often they are told of its necessity!

In follow-up exercises, at this stage in their development, the students would be required to submit answers arrived at through manual work. However, they are encouraged to check their answers to more difficult examples with DERIVE.

We do not treat the limit definition of a derivative very formally at the beginning of our work in calculus. For these students it would prove an obstacle to their understanding. It is given during a lecture but we do not require the students to work with it initially. Later, once they have begun to use calculus successfully to solve simple problems, we can return to the limit definition and use it to discover the derivatives of, for example, sinx and lnx and to consider some of the standard limits like sinx/x, as can be seen in figures (viii), (ix) and (x).

EXPONENTIAL AND LOGARITHM FUNCTIONS

These functions are absolutely essential for engineers and so in our course we introduce them as quickly as possible. After students have become familiar with the processes of differentiation and integration we embark on a study of exponential and logarithm functions.

DERIVE's excellent graphical and split-window facilities allow for an interesting investigation of these functions either in the laboratory or the classroom. The general behaviour of exponential growth and decay functions are quickly established by plotting the graphs of, for example,

2^x	6^x	10^x	e^x
2^{-x}	6^{-x}	10^{-x}	e^{-x}.

Indeed, the number e is encountered here for the first time as the base whose exponential graph lies between 2^x and 6^x, and is closer to 2^x (figures (xi) and (xii)). Its actual value is discussed later.

Also the beautiful symmetry between growth and decay graphs can be easily demonstrated, as shown in figure (xiii).

Logarithm functions to different bases are dealt with next. DERIVE has a special code for these functions. Instead of $\log_2 x$, for example, DERIVE needs the input log(x,2). We show the general behaviour of logarithm functions by plotting the graphs of:

log(x,2) log(x,6) log(x,10) log(x,e).

See figure (xiv) .

log(x,e) is shown to be the function "natural logarithm" by plotting the expression lnx and watching this graph overdraw the previous one.

At this point the inverse nature of exponential and logarithm functions is introduced. The students are already aware that e^x and lnx are activated by the same key on their calculator. The first one is usually activated with the "inv" or "shift" key. (They may not have used them before but their working knowledge of their calculators should make this evident.) Similarly they know that 10^x and logx work in the same way. By plotting e^x and lnx, 10^x and log(x,10) on the same grid DERIVE enables them to see that an exponential function and its corresponding logarithm function are reflections of each other in the line y = x (figure (xv).

DIFFERENTIAL EQUATIONS

A major application of calculus for engineers is differential equations. One of the intentions of the HITECC program is to introduce students to differential equations as quickly as possible.

We begin with word problems which have to be translated into mathematical expressions and these expressions are first order differential equations (figure (xvi)). Earlier in the course the students would have had experience of doing this with word problems which led to simultaneous equations or quadratic equations. This particular skill of translating words into symbols is very important and is always something our students find quite hard. Having DERIVE available to do the manipulation enables more time to be spent on developing these translation skills.

Once a differential equation has been constructed consideration is given to its solution. At this stage we only present the students with those differential equations which can be solved by direct integration or by the separation of variables.

Suppose we are trying to solve the ODE in figure (xvi). We avoid the DERIVE utility file about ODEs to begin with and actually create the necessary integrals in the following way:

Options Input Word;
Author $1/(\theta - 15)$; (Expression #1)

Calculus Integrate Expression #1 **Variable** θ;

Lower limit θ_0 **Upper limit** θ; (Expression #2)

Author -4/45; (Expression #3)
Calculus Integrate Expression #3 **Variable** t;
Lower limit t_0 **Upper limit** t; (Expression #4)
Build Expression #2 = Expression #4); (Expression #5)
Simplify Expression #5; (Expression #6)
Manage Substitute Expression #6
$t_0 = 0$ $\theta_0 = 200$; (Expression # 7)

Simplify Expression #7; (Expression #8)
soLve Expression #8 **Variable** θ; (Expression #9)

Plot;
Scale x:10 y:100;
Plot Expression #9;
Algebra;
Manage Substitute Expression #9 t = 10; (Expression #10)
approX Expression #10.

The solution of the problem is:

$$\theta = 185e^{-4t/45} + 15$$

which appears as expression #9. Substituting $t = 10$ in this expression yields, as expression #11, the value:

$$\theta = 91.06^{o}C \quad (2DP).$$

The interaction with DERIVE is shown in figures (xvii) and (xviii).

CONCLUSION

In approximately twelve weeks, 72 hours of teaching, these HITECC students have come from a situation in which they knew no advanced mathematics whatsoever to one in which they can solve simple examples of differential equations. During this period approximately 36 hours of the teaching will have been devoted to calculus and 14 of these hours will have been spent using DERIVE. There is a further term of 12 weeks during which we do more mathematics. What has been described above is focussed on the way in which we introduce calculus.

We find that DERIVE helps to stimulate the students interest in mathematics. It takes the drudgery out of algebraic manipulation and allows us to concentrate on concepts. Students dare to experiment in the laboratory and ask, *"What happens if....?"* They can actually find out for themselves how the graph of a function changes as a parameter is altered; they can draw so many curves of a particular type so quickly that they begin to "feel" what the mathematics is doing.

A certain sensitivity is required, however. DERIVE by itself will not teach. One of its drawbacks is the immediacy of its response in certain situations like integrating a complicated function and presenting the student with an equally complicated answer that the student does not understand. It is the interaction of the student with the package, the teacher and other students that creates a wonderful environment in which teaching and learning can take place. Handled like this, DERIVE certainly lives up to its description as "A Mathematical Assistant".

Last academic year, 1990/91, 63 students completed our course. Of these, 36 students are now on degree courses and 26 are on Higher National Diploma courses in engineering. Only one student failed.

Details of the student investigations have been published as *"DERIVE-based Investigations for Post-16 Core Mathematics"* Watkins AJP, (Chartwell-Bratt 1992).

Figure (i) - Getting Started with *DERIVE*

Manage Substitute $(a + 5b)^3 - 12a^2b^5$ $\quad a = 9.5 \;\; b = -7.02$

$$x - \frac{(x^3 - 4x^2 + 7x + 1)}{(3x^2 - 8x + 7)} \qquad x = 0.5$$

Factor $4p^2 - 23p + 15$

$2x^3 + x^2 - 13x + 6$

$w^4 + 10w^3 + 13w^2 - 96\,w - 180$

Expand $(3a - 7)^5$

$(3p - 8)(2p^3 + 11p^2 + 7p - 1)$

soLve $9(4r + 7) - 5(r + 2) = 8(3r + 1) - (2r - 5)$

$2t^2 + 17t - 63 = 0$

$3a^3 + 2a^2 = 61a - 20$

$6x^4 - 5x^3 - 14x^2 - x + 2 = 0$

Figure (ii) - Differentiating positive powers

8: x^5

9: $\frac{d}{dx} x^5$

10: $5\,x^4$

11 x^6

12: $\frac{d}{dx} x^6$

13: $6\,x^5$

```
COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage
         Options Plot Quit Remove Simplify Transfer moVe Window approX
Compute time: 0.0 seconds
Simp(12)                                  Free:100%          Derive Algebra
```

Figure (iii) - Differentiating negative powers

14: x^{-5}

15: $\frac{d}{dx} x^{-5}$

16: $-\frac{5}{x^{6}}$

17: x^{-6}

18: $\frac{d}{dx} x^{-6}$

19: $-\frac{6}{x^{7}}$

```
COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage
         Options Plot Quit Remove Simplify Transfer moVe Window approX
Compute time: 0.0 seconds
Simp(18)                                Free:100%              Derive Algebra
```

Figure (iv) - Differentiating fractional powers

20: $x^{1/2}$

21: $\frac{d}{dx} x^{1/2}$

22: $\frac{1}{2\sqrt{x}}$

23: $x^{-3/4}$

24: $\frac{d}{dx} x^{-3/4}$

25: $-\frac{3}{4x^{7/4}}$

```
COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage
         Options Plot Quit Remove Simplify Transfer moVe Window approX
Compute time: 0.1 seconds
Simp(24)                                Free:100%              Derive Algebra
```

Figure (v) - Integrating positive powers

$$x^5$$

12: $\int x^5 \, dx$

13: $\dfrac{x^6}{6}$

14: x^6

15: $\int x^6 \, dx$

16: $\dfrac{x^7}{7}$

```
COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage
         Options Plot Quit Remove Simplify Transfer moVe Window approX
Compute time: 0.1 seconds
Simp(15)                                  Free:100%              Derive Algebra
```

Figure (vi) - Integrating negative powers

17: x^{-3}

18: $\int x^{-3} \, dx$

19: $\dfrac{1}{2\,x^2}$

20: x^{-4}

21: $\int x^{-4} \, dx$

22: $-\dfrac{1}{3\,x^3}$

```
COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage
         Options Plot Quit Remove Simplify Transfer moVe Window approX
Compute time: 0.1 seconds
Simp(21)                                  Free:100%              Derive Algebra
```

Figure (vii) - Integrating fractional powers

23: $x^{1/2}$

24: $\int x^{1/2}\, dx$

25: $\dfrac{2\, x^{3/2}}{}$

26: $x^{-1/3}$

27: $\int x^{-1/3}\, dx$

28: $\dfrac{3\, x^{2/3}}{2}$

```
COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage
         Options Plot Quit Remove Simplify Transfer moVe Window approX
Compute time: 0.1 seconds
Simp(27)                                  Free:100%              Derive Algebra
```

Figure (viii) - The equivalence of "lim h -> 0" and "d/dx"

1

1: $\dfrac{\mathrm{SIN}\,(x + h) - \mathrm{SIN}\,(x)}{h}$

2: $\lim\limits_{h \to 0} \dfrac{\mathrm{SIN}\,(x + h) - \mathrm{SIN}\,(x)}{h}$

3: $\mathrm{COS}\,(x)$

4: $\dfrac{\mathrm{LN}\,(x + h) - \mathrm{LN}\,(x)}{h}$

5: $\lim\limits_{h \to 0} \dfrac{\mathrm{LN}\,(x + h) - \mathrm{LN}\,(x)}{h}$

6: $\dfrac{1}{x}$

2

1: $\mathrm{SIN}\,(x)$

2: $\dfrac{d}{dx}\,\mathrm{SIN}\,(x)$

3: $\mathrm{COS}\,(x)$

4: $\mathrm{LN}\,(x)$

5: $\dfrac{d}{dx}\,\mathrm{LN}\,(x)$

6: $\dfrac{1}{x}$

```
COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage
         Options Plot Quit Remove Simplify Transfer moVe Window approX
Compute time: 0.1 seconds
Simp(5)                                   Free:99%               Derive Algebra
```

Figure (ix) - The limit of sinx/x as x -> 0

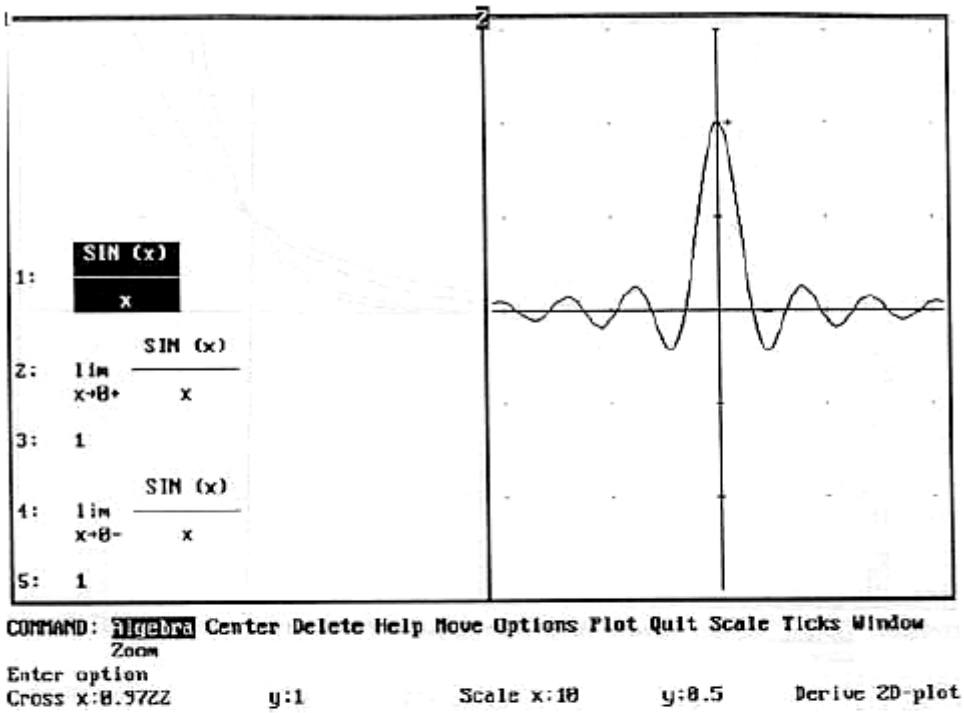

Figure (x) - The limit of $\sin x/x^2$ as x -> 0

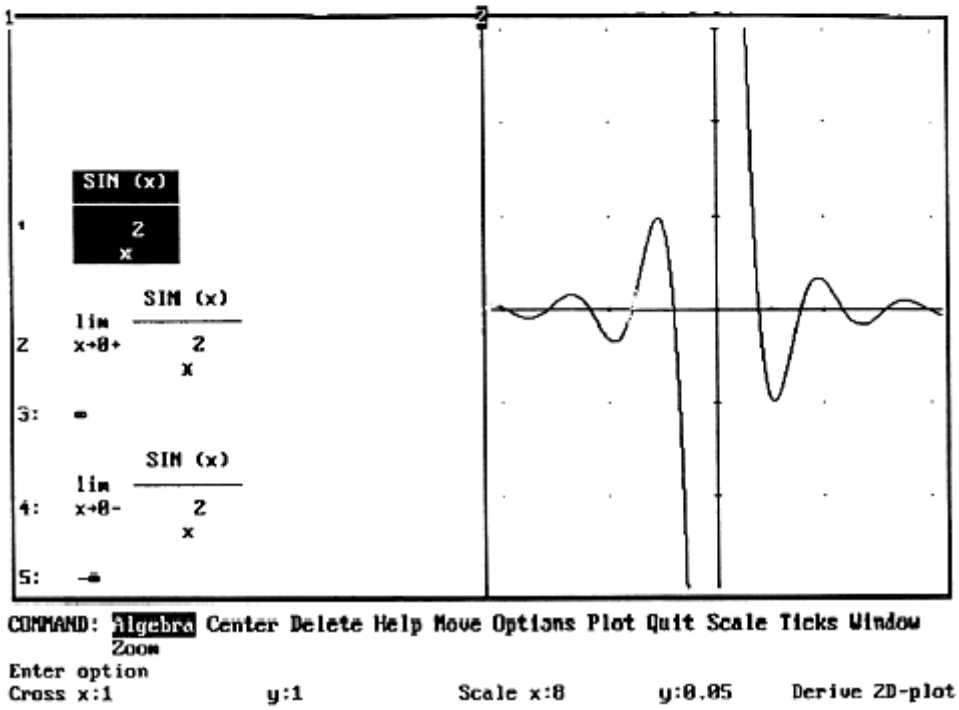

Figure (xi) - Graphs showing exponential growth

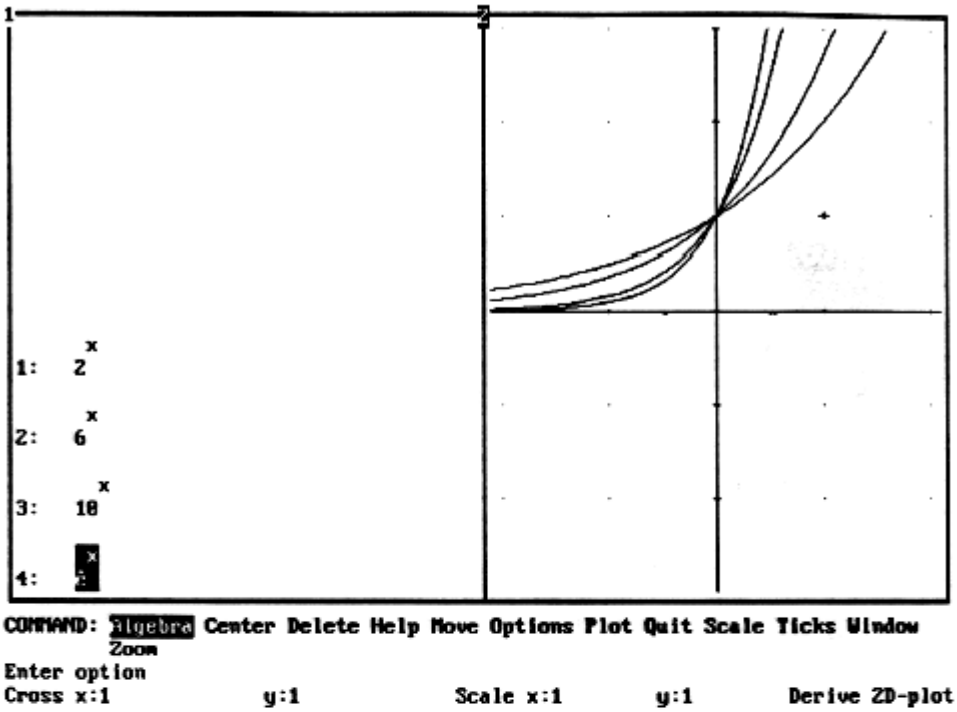

Figure (xii) - Graphs showing exponential decay

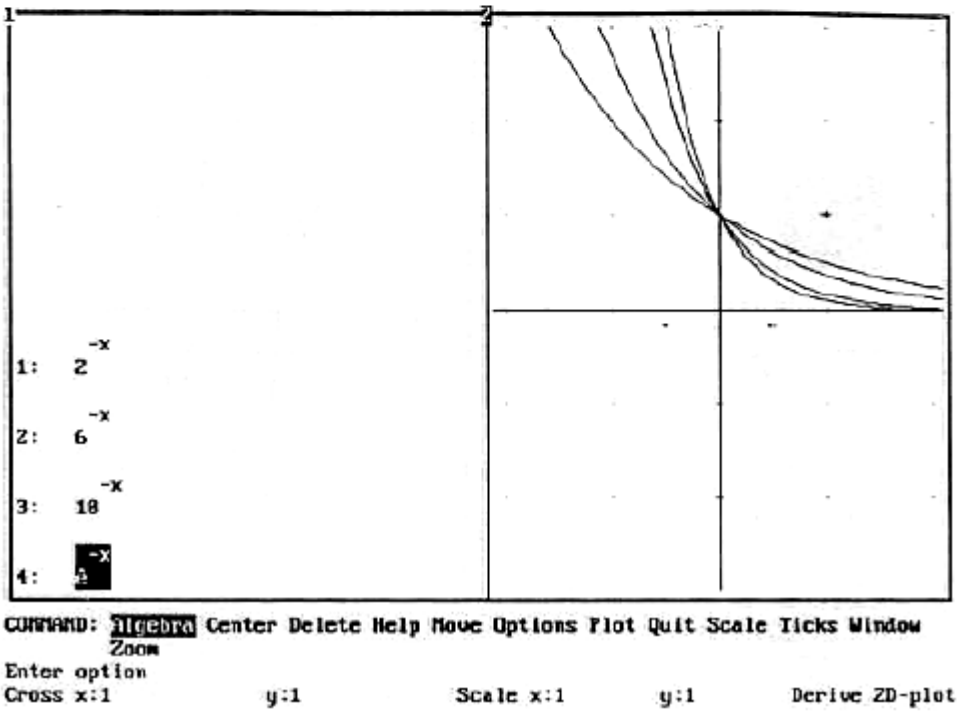

Figure (xiii) - The symmetry of growth and decay graphs

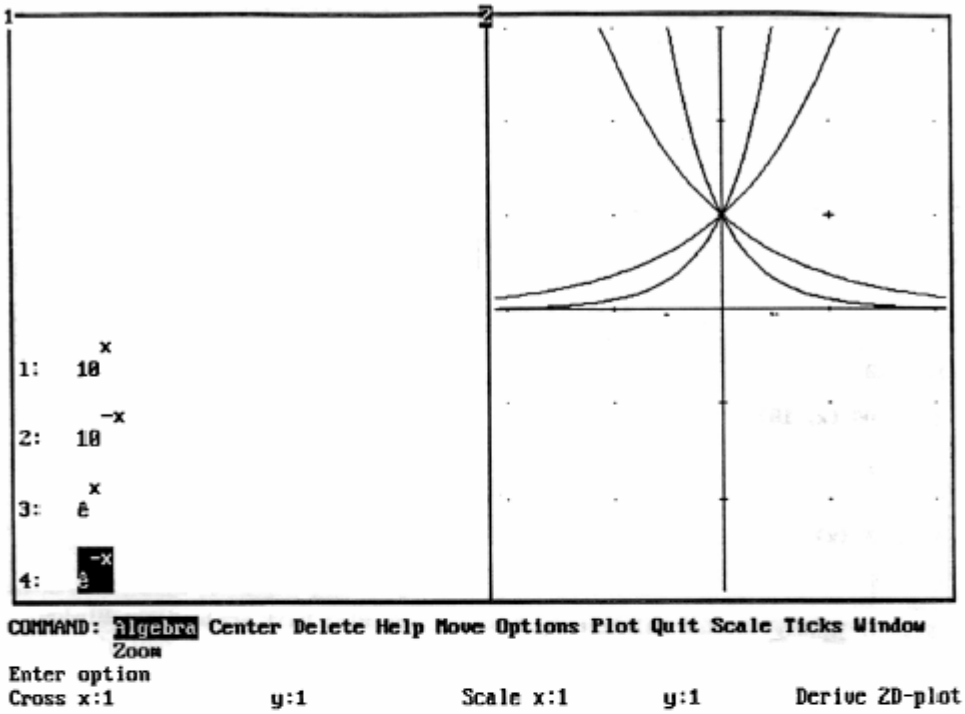

Figure (xiv) - Graphs of logarithm functions

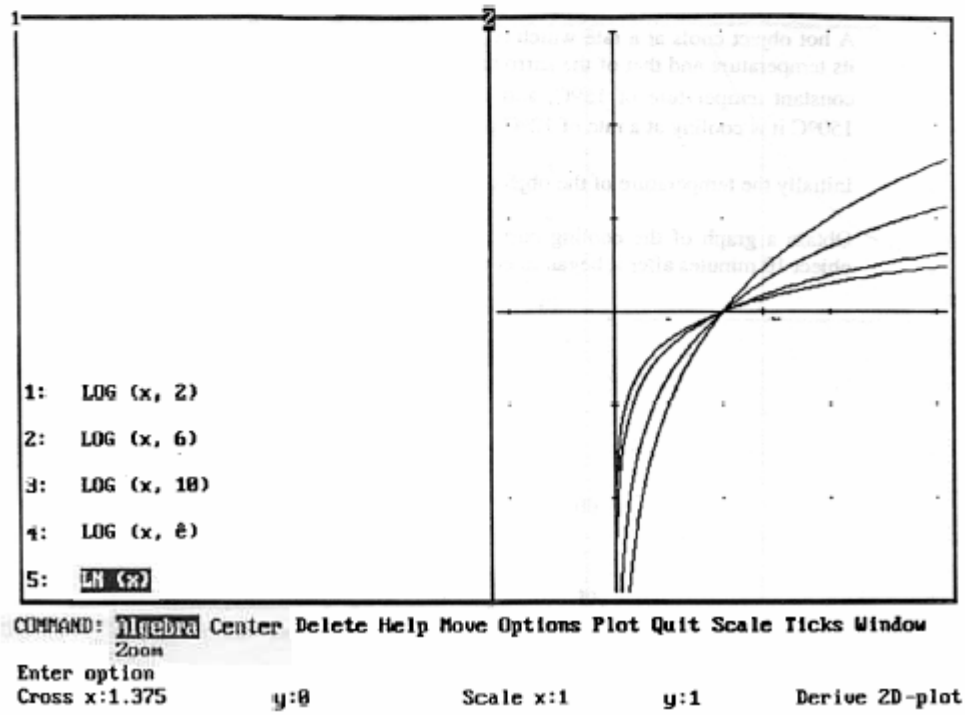

Figure (xv) - Exponential and logarithm as inverse functions

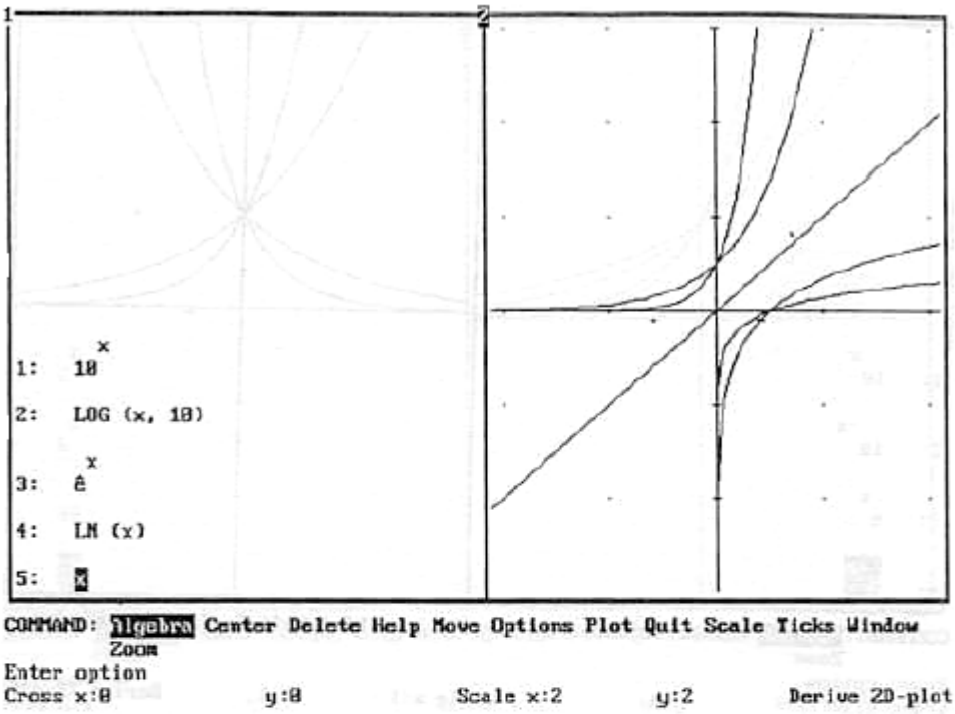

Figure (xvi) - A typical differential equation problem

A hot object cools at a rate which is proportional to the difference between its temperature and that of the surrounding air. The surroundings remain at a constant temperature of 15°C, and when the temperature of the object is 150°C it is cooling at a rate of 12°C per minute.

Initially the temperature of the object is 200°C.

Obtain a graph of the cooling curve and determine the temperature of the object 10 minutes after it began to cool.

$$\frac{d\theta}{dt} = -\frac{4}{45}(\theta - 15)$$

Figure (xvii) - Solving the differential equation problem

1: $\dfrac{}{\theta - 15}$

2: $\displaystyle\int_{\theta 0}^{\theta} \frac{1}{\theta - 15}\, d\theta$

3: $-\dfrac{4}{45}$

4: $\displaystyle\int_{t0}^{t} \frac{4}{45}\, dt$

5: $\displaystyle\int_{\theta 0}^{\theta} \frac{1}{\theta - 15}\, d\theta = \int_{t0}^{t} -\frac{4}{45}\, dt$

COMMAND: Author Build Calculus Declare Expand Factor Help Jump soLve Manage Options Plot Quit Remove Simplify Transfer moVe Window approX

Free:100% Derive Algebra

Figure (xviii) - Solving the differential equation problem

6: $LN(\theta - 15) - LN(\theta 0 - 15) = \dfrac{4\, t0}{45} - \dfrac{4\, t}{45}$

7: $LN(\theta - 15) - LN(200 - 15) = \dfrac{4\, \theta}{45} - \dfrac{4\, t}{45}$

8: $LN\left[\dfrac{\theta - 15}{185}\right] = -\dfrac{4\, t}{45}$

9: $\theta = 185\, \hat{e}^{-4\, t / 45} + 15$

10: $\theta = 185\, \hat{e}^{-4\, 10 / 45} + 15$

11: $\theta = 91.0557$

COMMAND: Algebra Center Delete Help Move Options Plot Quit Scale Ticks Window Zoom

Enter option

Cross x:10 y:90.625 Scale x:10 y:100 Derive 2D-plot